

Reliability Assessment with Correlated Variables using Support Vector Machines

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This paper presents an approach to estimate probabilities of failure in cases where the random variables are correlated. An explicit limit state function is constructed in the uncorrelated standard normal space using the Nataf transformation and a support vector machine (SVM). The method of explicitly constructing the limit state function is referred to as explicit design space decomposition (EDSD), which also includes an adaptive sampling strategy to build an accurate SVM approximation. Several analytical examples with various distributions and also multiple failure modes are presented.

I. Introduction

In reliability, the vast majority of studies use uncorrelated random variables to represent uncertainties. However, in many applications (such as biomechanical problems), random variables associated with a system are often correlated. Not accounting for such correlation can lead to erroneous probability of failure estimates. Although there are techniques to calculate such a probability of failure for specific cases (e.g., normal distributions with inexpensive function evaluations), there is a need for approaches that can handle highly nonlinear limit state functions and arbitrary probabilistic distributions while maintaining a reasonable number of function evaluations.

The general approach for addressing problems with correlated random variables is to convert them into uncorrelated standard Gaussian variables. Two of the common methods are Rosenblatt¹ and Nataf^{2,3} transformations. The difficulty with implementing Rosenblatt transformation lies in its requirement of the knowledge of the joint probability density function, which is not always available. In addition, the transformation is not invariant with respect to the order by which the variables are transformed. The Nataf transformation, used in this work, only requires the marginal probability density functions, and is therefore easier to implement. The Nataf transformation is explained in detail in Section III.

In the literature, most articles dealing with correlated random variables in reliability assessment use moment-based methods such as First and Second Order Reliability Methods (FORM and SORM)⁴ It is, however, known that accuracy of such methods is hampered if the limit state is highly nonlinear. Simulation-based methods, such as Monte Carlo Sampling⁵ in the uncorrelated space have also been used, but application of such methods is not practical in general due to the large number of samples.⁶ In addition, when dealing with correlated variables, the MCS samples generated in the standard Gaussian space need to be transformed back into the original space to evaluate the responses.

In this work in progress, a general approach to calculate probabilities with correlated variables while limiting the computational cost is proposed. This is achieved by replacing the threshold-level contour of the actual limit state function with by a Support Vector Machine (SVM) classifier in the uncorrelated standard normal space.^{7,8,6} This method of constructing an explicit approximation of the failure boundary is referred to as Explicit Design Space Decomposition (EDSD).^{9,6} In order to obtain an accurate SVM and limit the number of function evaluations, an adaptive sampling technique is used.^{10,11} To determine the “class” (failure or safe), each sample used for the construction of the SVM is mapped “back” to the original correlated

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space using an inverse Nataf Transformation. Because each sample requires a function evaluation (e.g., a finite element analysis), their number must be limited. Once an SVM is obtained, MCS is performed in the uncorrelated standard Gaussian space based on the approximation.⁶ It should be noted that the MCS samples do not require inverse transformation to the original space because the explicit SVM limit state function in the uncorrelated space is available. In addition, the proposed approach introduces the probability of misclassification by SVM using Probabilistic Support Vector machines (PSVMs) in the probability of failure estimate. This leads to a relatively conservative failure probability estimate.¹²

The use of SVM is appealing due to several reasons. It is based on the classification of the space into a safe and failure domains. Construction of the boundary depends only on the class of samples instead of the actual response values. Therefore, the method is applicable to binary (pass or fail) problems¹³ or problems with discontinuities.^{9,10,14} In addition, SVM presents a natural way to handle multiple failure modes especially if the system considered is serial since a sample would be considered failed if any one mode is active.

The paper is organized as follows. Section II describes the basic methodology. Section III presents the Nataf transformation along with the Jacobian of the transformation. Section IV to VI provide the basics of ESD, SVM, and adaptive sampling. Section VII introduces a relatively conservative failure probability estimate based on Probabilistic Support Vector machines (PSVMs).¹² Finally, Section VIII provides several examples of analytical problems with various probabilistic distributions.

II. Basic methodology

The basic methodology for calculating probability of failure with correlated variables, using Support Vector Machines (SVMs) (Section V),^{7,8} is presented in this section. The generation of MCS samples directly in the correlated space (X-space) is possible only for specific distribution types, such as Gaussian or Truncated Gaussian.

In order to generalize the calculation of probabilities of failure for arbitrary distributions, this work proposes to construct an SVM-based limit state function in the uncorrelated Standard Gaussian space (U-space). The probability of failure is then calculated in using MCS in the U-space.

The basic procedure for constructing a limit state approximation in the U-space is demonstrated in Figure 1. The main steps are:

- Generation of an initial uniform design of experiments (DOE)^{15,16,17} in uncorrelated standard Gaussian space (U-space).
- Inverse Nataf transformation¹⁸ (Section III) of the DOE samples to the original correlated space (X-space) for response evaluation and class (fail or safe) assignment.
- Construction of an initial approximation of the limit state in the U-space using explicit design space decomposition (ESD) (Section IV).
- Calculation of initial probability of failure estimate based on the SVM approximation, using MCS in uncorrelated space. Probability of failure is calculated as:

$$P_f = \frac{N_f}{N_{MC}} = \frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} I_g(\mathbf{u}_i) \quad (1)$$

where \mathbf{u}_i represents Monte Carlo samples in U-space, N_{MC} is total number of Monte Carlo samples and N_f is number of Monte Carlo samples lying in failure region based on the SVM approximation. $I_g(\mathbf{u})$ is an indicator function that is 0 for samples in the safe region and 1 for those in the failure region:

$$I_g(\mathbf{u}) = \begin{cases} 1 & s(\mathbf{u}) \leq 0 \\ 0 & s(\mathbf{u}) > 0 \end{cases} \quad (2)$$

- Refinement of the limit state approximation using adaptive sampling.¹¹ Each additional sample is transformed back to the correlated space to obtain its response. The probability of failure estimate is updated after each sample evaluation.

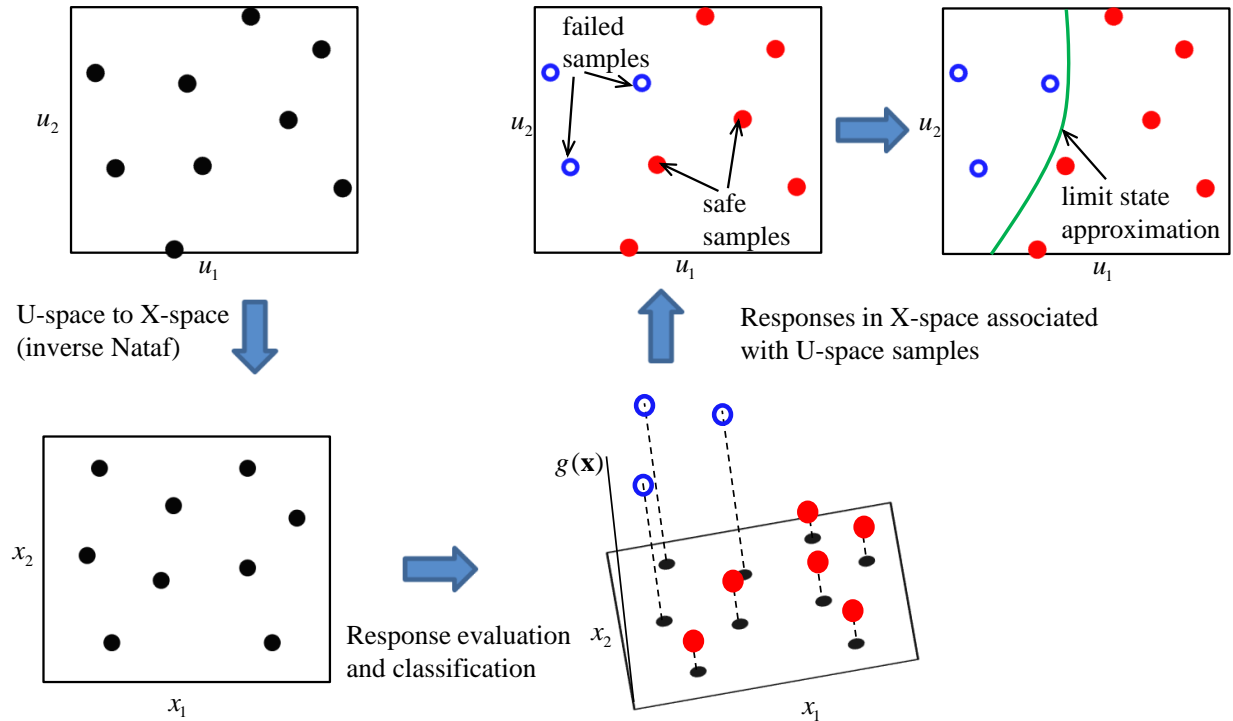


Figure 1. Summary of the method for limit state approximation in U-space.

III. Nataf transformation

A. Basic principle

The Nataf transformation is performed in two steps. The first step transforms the variables of any arbitrary distributions to correlated standard normal variables. The second step transforms the latter into uncorrelated standard normal variables.

1. Step 1

Consider n correlated random variables X_i whose marginal cumulative distribution functions, F_{X_i} and correlation matrix ρ are known. The first transformation generates correlated standard normal variables U_i^0 , with correlation matrix ρ^0 , obtained as follows:^{18,5}

$$\begin{aligned}
 \Phi(u_1^0) &= F_{X_1}(x_1) \\
 \Phi(u_2^0) &= F_{X_2}(x_2) \\
 &\vdots \\
 \Phi(u_n^0) &= F_{X_n}(x_n)
 \end{aligned} \tag{3}$$

where Φ is the cumulative standard normal distribution.

One of the critical aspects of the Nataf transformation is to relate the correlation coefficients ρ_{ij} and ρ_{ij}^0 . For this purpose, we use the fact that the joint density function of two correlated variables X_i and X_j can be expressed as a function of the joint normal probability density function of the correlated variables U_i^0 and U_j^0 using the following equation:

$$f_{X_i X_j}(x_i, x_j) = \phi_2(u_i^0, u_j^0, \rho_{ij}^0) \frac{f_{X_i}(x_i) f_{X_j}(x_j)}{\phi(u_i^0) \phi(u_j^0)} \quad (4)$$

where $\phi_2(u_i^0, u_j^0, \rho_{ij}^0)$ is the bivariate standard normal probability density function with correlation coefficient of ρ_{ij}^0 . The expression of ϕ_2 is:

$$\phi_2(u_i^0, u_j^0, \rho_{ij}^0) = \frac{1}{2\pi\sqrt{1 - (\rho_{ij}^0)^2}} \exp \left[-\frac{(u_i^0)^2 - 2\rho_{ij}^0 u_i^0 u_j^0 + (u_j^0)^2}{\phi(u_i^0) \phi(u_j^0)} \right] \quad (5)$$

The relation between ρ and ρ^0 is derived as follows:

$$\begin{aligned} \rho_{ij} &= \frac{\text{cov}(X_i, X_j)}{\sigma_{X_i} \sigma_{X_j}} \\ &= \frac{E(X_i X_j) - E(X_i)E(X_j)}{\sigma_{X_i} \sigma_{X_j}} \\ &= \frac{1}{\sigma_{X_i} \sigma_{X_j}} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_i x_j f_{X_i, X_j}(x_i, x_j) dx_i dx_j - \int_{-\infty}^{+\infty} x_i f_{X_i}(x_i) dx_i \int_{-\infty}^{+\infty} x_j f_{X_j}(x_j) dx_j \right) \end{aligned} \quad (6)$$

Using $f_{X_i}(x_i) dx_i = \phi(u_i^0) du_i^0$, the double integral can be simplified as:

$$\begin{aligned} \rho_{ij} &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u_i^0 u_j^0 f_{X_i, X_j}(x_i, x_j) dx_i dx_j \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u_i^0 u_j^0 \phi_2(u_i^0, u_j^0; \rho_{ij}^0) du_i^0 du_j^0 \end{aligned} \quad (7)$$

where $u_i^0 = (x_i - \mu_i)/\sigma_i$ and $u_j^0 = (x_j - \mu_j)/\sigma_j$. We need to solve the double integral to find the corresponding ρ_{ij}^0 for a given ρ_{ij} . In engineering applications, this process was often replaced by polynomial approximation¹⁹ in order to remove the need to solve the double integral. For instance, consider a two variable problem with uniform-exponential variables and correlation coefficient equal to 0.2. In this case, the relation is approximated as:

$$\begin{aligned} \rho_{12}^0 &= R \rho_{12} \\ R &= 1.133 + 0.029 \rho_{12}^2 \end{aligned} \quad (8)$$

Although, these relations can be very accurate, it is nowadays possible to solve the double integral numerically in an efficient manner. This latter approach has been used in this paper.

2. Step 2

Once all the elements of the ρ_0 correlation matrix are found, the second step of the Nataf Transformation consists of decorrelating the variables to obtain uncorrelated standard normal variables U_i .^{18,20} The following linear transformation is used:

$$U^0 = AU + B \quad (9)$$

U^0 and U here are vectors containing the correlated and uncorrelated standard normal variables respectively. A and B are matrices. By taking the mean of both sides, we obtain:

$$E(U^0) = E(AU + B) = AE(U) + B = 0 \quad (10)$$

Because both variables are standard normal, we obtain $B = 0$. A is related to the correlation matrix ρ_0 with the following equation:

$$AA^T = \rho^0 \quad (11)$$

This relation can be rather simply proven by the fact that $\rho_{ij}^0 = E(u_i^0 u_j^0)$. Therefore, A can be obtained by Cholesky decomposition of ρ^0 , which is a positive definite matrix. For a two dimensional problem:

$$A = \begin{bmatrix} 1 & 0 \\ \rho_{12}^0 & \sqrt{1 - (\rho_{12}^0)^2} \end{bmatrix} \quad (12)$$

For a three variable problem:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{12}^0 & \sqrt{1 - (\rho_{12}^0)^2} & 0 \\ \rho_{13}^0 & \frac{\rho_{23}^0 - \rho_{13}^0 \rho_{12}^0}{\sqrt{1 - (\rho_{12}^0)^2}} & \sqrt{1 - (\rho_{13}^0)^2 - \frac{(\rho_{23}^0 - \rho_{13}^0 \rho_{12}^0)^2}{1 - (\rho_{12}^0)^2}} \end{bmatrix} \quad (13)$$

B. Jacobian

The Jacobian matrix¹⁸ defining the mapping between the original correlated space to the uncorrelated space is defined by its general term as: $\partial u_j / \partial x_i$. Using the chain rule and the relation between dx_i and du_i^0 :

$$dx_i = \frac{\phi(u_i^0)}{f_{X_i}(x_i)} du_i^0 \quad (14)$$

The Jacobian is:

$$J = \text{diag} \left[\frac{\phi(u_i^0)}{f_{X_i}(x_i)} \right] A \quad (15)$$

As an example, the determinant of the Jacobian for a two dimensional problem with exponential-exponential variables, is depicted in Figure 2. The Figure shows how a grid will transform through the Nataf Transformation. For this case, the determinant of the Jacobian is always positive and thus the mapping between the X-space and the U-space is unique. The small values close to the origin of the X-space lead to large distances in the U-space.

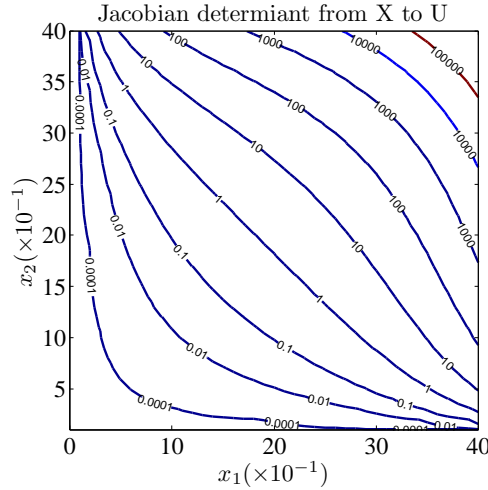


Figure 2. Contour of Jacobian determinant from X to U space.

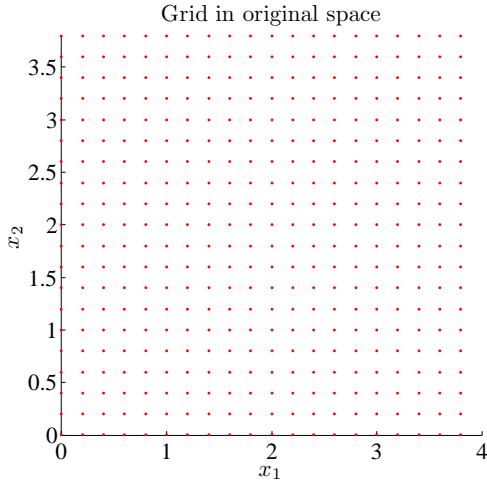


Figure 3. Grid in X-space

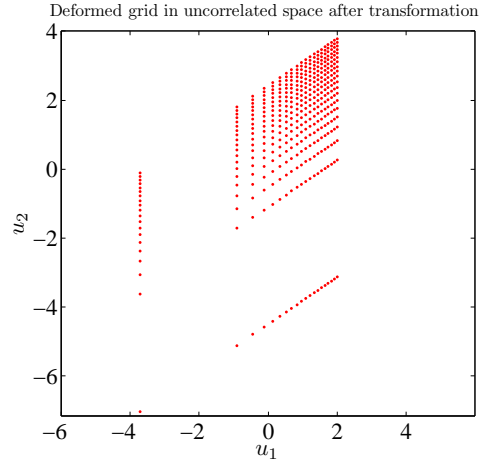


Figure 4. Grid in U-space after transformation

IV. Explicit Design Space Decomposition (EDSD)

Explicit Design Space Decomposition is a classification-based method that can be used for constructing approximations of limit state functions. The basic idea in EDSD^{9,6} is to explicitly decompose the space into regions of distinct behavior, e.g. safe and failed. Instead of approximation of responses, EDSD is based only on their classification. As a result, the method has several advantages, such as handling of discontinuous and binary responses,^{9,6,13,14} and multiple failure modes.^{21,22} The main steps of EDSD are as follows.

- Generation of an initial design of experiments (DOE). In this article samples are selected in the uncorrelated space using Centroidal Voronoi Tessellations (CVT).^{16,17}
- Response evaluation at the DOE samples.
- Classification of the samples as safe (+1) or failed (-1) based on a threshold responses or using clustering.
- Construction of an explicit boundary separating the +1 and -1 classes. In this article, SVMs are used to approximate failure boundaries.

V. Support Vector Machines (SVMs)

SVMs^{7,8} belong to a class of machine learning techniques that can be used for both regression and classification. In the context of EDSD, SVMs are used for classification. SVMs have gained popularity in the computer science community due to their good generalization capability in classification. In the context of reliability assessment also, flexibility of SVMs in approximating highly nonlinear limit state functions has been shown in several papers.^{23,6,10,11,22} In the case of multiple failure modes, a single SVM can be used to represent all the modes. The basic theory of SVMs for binary classification is presented briefly in this section.

An SVM is used to construct an explicit equation that separates samples belonging to two classes labeled as +1 and -1. Given a set of N training samples \mathbf{u}_i in a n -dimensional U-space, and the corresponding class labels, an SVM boundary is given as:

$$s(\mathbf{u}) = b + \sum_{i=1}^N \lambda_i y_i K(\mathbf{u}_i, \mathbf{u}) = 0 \quad (16)$$

where b is a scalar referred to as the bias, λ_i are Lagrange Multipliers obtained from the quadratic programming optimization problem used to construct the SVM, and K is a kernel function. The classification of any

arbitrary point \mathbf{u} is given by the sign of $s(\mathbf{u})$. The training samples for which the Lagrange Multipliers are non-zero are referred to as the *support vectors*. The number of support vectors NSV is usually much smaller than N , and therefore, only a small fraction of the samples affects the SVM equation. The kernel function K in Equation 16 can have several forms, such as polynomial or Gaussian radial basis kernel. The Gaussian kernel (Equation 17) is used in this article.

$$K(\mathbf{u}_i, \mathbf{u}) = \exp\left(-\frac{\|\mathbf{u}_i - \mathbf{u}\|^2}{2\sigma^2}\right) \quad (17)$$

where σ is the width parameter of the Gaussian kernel. It should be noted that although the SVM equation is presented for U-space in terms of \mathbf{u} , it is valid for any space. In this article, SVMs are constructed in U-space.

VI. Adaptive Sampling

EDSD using a static DOE may not be sufficient for predicting accurate failure probabilities, unless a large number of samples N is used. Therefore, to obtain high accuracy with limited samples, an initially sparse DOE in U-space is adaptively updated. The initial DOE is used to construct the first SVM approximation of the limit state function. This gives an initial idea of regions that are important for refining the SVM approximation. Additional samples are then added iteratively to update the SVM boundary. The SVM update algorithm consists of two types of samples:

- Primary sample
- Secondary samples

A primary sample is selected such that it lies on the SVM boundary in a sparsely populated region. Such a point lying near the boundary has high probability of misclassification. It is selected by maximizing the distance to the closest training sample:

$$\begin{aligned} \max_{\mathbf{u}} \quad & \|\mathbf{u} - \mathbf{u}_{nearest}\| \\ \text{s.t.} \quad & s(\mathbf{u}) = 0 \end{aligned} \quad (18)$$

The objective function in Equation 18 is not differentiable everywhere in the space. The problem is reformulated to make it differentiable:

$$\begin{aligned} \max_{\mathbf{u}, z} \quad & z \\ \text{s.t.} \quad & \|\mathbf{u} - \mathbf{u}_i\| \geq z \quad i = 1, 2, \dots, N \\ & s(\mathbf{u}) = 0 \end{aligned} \quad (19)$$

A secondary sample is selected in a region in the vicinity of SVM boundary that has the highest local unbalance of data from the two classes. Unbalance of data is quantified as $|d_- - d_+|$, where d_- and d_+ are the distances to the closest -1 and $+1$ samples. A high unbalance of data may be an indicator of a phenomenon known as “locking” of SVM, which may result in a very low rate of convergence of SVM to the actual limit state function. Although change in SVM due to primary samples may be negligible in such cases, secondary samples are useful in removing locking. This is shown conceptually in Figure 5. Selection of a secondary sample is a two step process:

- Selection of a point \mathbf{u}_c on $s(\mathbf{u}) = 0$ with highest measure of unbalance.

$$\begin{aligned} \max_{\mathbf{u}} \quad & (d_-(\mathbf{u}) - d_+(\mathbf{u}))^2 \\ \text{s.t.} \quad & s(\mathbf{u}) = 0 \end{aligned} \quad (20)$$

- Selection of secondary sample within a hypersphere centered at \mathbf{u}_c .

$$\begin{aligned} \min_{\mathbf{u}} \quad & \text{sign}(d_-(\mathbf{u}_c) - d_+(\mathbf{u}_c))s(\mathbf{u}) \\ & \|\mathbf{u} - \mathbf{u}_c\| - \frac{1}{4}|d_-(\mathbf{u}) - d_+(\mathbf{u})| \leq 0 \end{aligned} \quad (21)$$

More details about SVM locking can be found in previous articles.^{11,22} A dedicated adaptive sampling method to calculate accurate failure probabilities using SVMs can also be found.²²

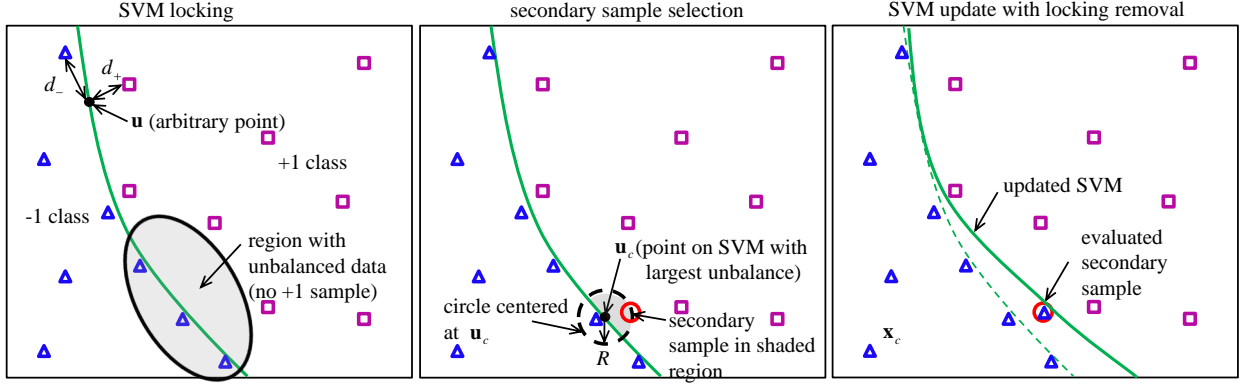


Figure 5. Locking of SVM with locally unbalanced data in the vicinity of SVM (left), selection of a secondary sample (mid), and update of SVM due to the evaluated secondary sample (right).

VII. Monte-Carlo Simulations based on Probabilistic Support Vector Machines (PSVMs)

Adaptive sampling (Section VI) increases the accuracy of SVM approximation and reduces the probability of misclassification. However, the error in the SVM prediction might result in an inaccurate probability of failure. Underestimation of probability of failure is especially hazardous as it may lead to an unsafe design. Therefore, it is useful to have a relatively conservative estimate of probability of failure to compensate for errors in SVM classification.

A. Conservative Failure Probability Estimate

A method based on PSVMs is used to provide a relatively conservative probability of failure compared to corresponding deterministic SVMs.¹² Unlike deterministic SVMs that provide binary classification, PSVMs provide a conditional probability of belonging to a specific class for any sample. Thus, for any Monte Carlo Sample lying in safe region or +1 class based on a deterministic SVM, the corresponding PSVM provides a probability of belonging to the -1 or failure class ($P(-1|\mathbf{u})$). The probability of failure given by Equation 1 is replaced by:

$$P_f^{PSVM} = \frac{1}{N_{MCS}} \left(\sum_{i=1}^{N_{MCS}} \gamma(-1|\mathbf{u}_i) \right),$$

$$\gamma(-1|\mathbf{u}) = \begin{cases} 1 & \mathbf{u}_i \in \Omega_f \\ 0 & \mathbf{u}_i \in \Omega_{sd} - \Omega_{misc} \\ P(-1|\mathbf{u}) & \mathbf{u}_i \in \Omega_{misc} \end{cases} \quad (22)$$

The difference between the traditional binary indicator function $I_g(\mathbf{u})$ (Equation 2) and $\gamma(-1|\mathbf{u})$ lies in their values in the region Ω_{misc} . Ω_{misc} is the region in safe domain, based on deterministic SVM, for which a probability of misclassification is considered in Equation 22. The contribution of MCS samples lying in Ω_{misc} to the probability of failure is zero in Equation 1. However, when using the PSVM-based method, these samples have a non-zero contribution in Equation 22. Therefore, Equation 22 provides a relatively conservative probability of failure. The region Ω_{misc} for considering the probability of misclassification by the SVM is:

$$\Omega_{misc} = \Omega_{sd} \cap \Omega (|s(\mathbf{u})| < 1 \cup s(\mathbf{u})(d_+(\mathbf{u}) - d_-(\mathbf{u})) \geq 0), \quad (23)$$

where Ω_{sd} is the safe domain based on the deterministic SVM, and $d_+(\mathbf{u})$ and $d_-(\mathbf{u})$ are the distances of \mathbf{u} to the closest +1 and -1 training samples in U-space. Ω_{misc} consists of two kinds of regions in the +1 class.

One is the SVM margin in the safe class and the other is the region with $d_+(\mathbf{u}) \geq d_-(\mathbf{u})$ (Figure 6). The calculation of $\gamma(-1|\mathbf{u})$ in the region Ω_{misc} , equal to $P(-1|\mathbf{u})$, is explained in the following section.

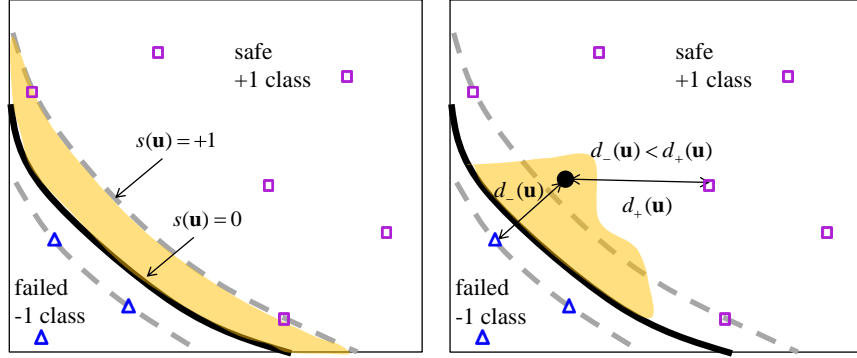


Figure 6. Definition of the region Ω_{misc} for considering the probability of misclassification. Ω_{misc} is the union of the two shaded regions in the left and the right figures.

B. Probabilistic Support Vector Machines (PSVMs)

Probabilistic support vector machines are used to map SVM values to conditional probabilities of belonging to a specific class. One of the commonly used PSVM models proposed by Platt involves the fitting of a sigmoid function using maximum likelihood.²⁴ This model, however, ignores the spatial distribution of samples. It only accounts for SVM values while calculating $P(+1|\mathbf{u})$ and $P(-1|\mathbf{u})$. To overcome this issue, a modified sigmoid model is used in this article that accounts for the spatial distribution of samples.²⁵ If \mathbf{u} is fixed in space, PSVM provides the probability of belonging to +1 or safe class:

$$P(+1|\mathbf{u}) = \frac{1}{1 + e^{As(\mathbf{u}) + B(\frac{d_-(\mathbf{u})}{d_+(\mathbf{u}) + \tau} - \frac{d_+(\mathbf{u})}{d_-(\mathbf{u}) + \tau)}}} \quad A < \frac{-3}{\min(s_{max}, -s_{min})}, B < 0, \quad (24)$$

where $d_-(\mathbf{u})$ and $d_+(\mathbf{u})$ are the distances to the closest -1 and +1 samples. τ is a small quantity (set equal to 10^{-100} in this work) added in order to avoid numerical issues at the evaluated training samples. s_{max} is the maximum SVM value among all training samples and s_{min} is the minimum SVM value among all training samples. The PSVM model in Equation 24 satisfies the following conditions:

- $P(+1|\mathbf{u}) \rightarrow 1$ if $s(\mathbf{u}) \rightarrow \infty$ or $d_+(\mathbf{u}) \rightarrow 0$
- $P(+1|\mathbf{u}) \rightarrow 0$ if $s(\mathbf{u}) \rightarrow -\infty$ or $d_-(\mathbf{u}) \rightarrow 0$
- $P(+1|\mathbf{u}) \rightarrow 0.5$ if $s(\mathbf{u}) \rightarrow 0$ and $d_-(\mathbf{u}) \rightarrow d_+(\mathbf{u})$

Parameters A and B of the PSVM model are calculated using maximum likelihood. For further details, the reader may refer to.²⁵ The conditional probability of belonging to -1 or failure class is:

$$P(-1|\mathbf{u}) = 1 - P(+1|\mathbf{u}) \quad (25)$$

VIII. Results

Several examples are presented in this section to demonstrate the calculation of probabilities of failure using correlated variables. To show the generality of the proposed approach, various distribution types, namely Gaussian, Exponential and Weibull, are used. The examples presented consist of two and three random variables. An example showing the use of PSVM for failure probability calculation is also presented. Probabilities of failure are calculated using 10^6 MCS samples for all the examples. In all of the following examples, $g(x) \leq 0$ represents failure domain. The following notations are used to present the results:

- P_f^{SVM} is the probability of failure predicted by SVM

- P_f^{PSVM} is the probability of failure predicted by PSVM
- P_f^{Ref} is the probability of failure by direct transformation
- P_f^{UC} is the probability of failure ignoring correlation
- 95% CI of P_f^{Ref} is 95% confidence interval of P_f^{Ref}
- ϵ^{SVM} is the relative error between P_f^{SVM} and P_f^{Ref}

$$\epsilon^{SVM} = \frac{P_f^{SVM} - P_f^{Ref}}{P_f^{Ref}} \times 100\% \quad (26)$$

- ϵ^{UC} is the relative error between P_f^{UC} and P_f^{Ref}

$$\epsilon^{UC} = \frac{P_f^{UC} - P_f^{Ref}}{P_f^{Ref}} \times 100\% \quad (27)$$

A. Two dimensional examples

Two examples with different distributions are presented in this section. The second example consists of four failure modes, all of which are represented using a single SVM. In both examples, the initial SVM in U-space is constructed with 30 samples. The SVM approximations are refined using adaptive sampling. The update is terminated if the relative change in probability of failure is less than 0.1% for five consecutive iterations.

1. Exponential-Exponential Distributions

The actual limit state function for this example, defined in X-space, is:

$$g(x_1, x_2) = 3 - (x_1 - 2)^3 - 3(x_2 - 1) = 0 \quad (28)$$

Both variables have exponential distributions with mean of 1 and the correlation coefficient is -0.4. Using the polynomial approximation¹⁹ to solve for ρ_{12}^0 ($R_{12} = 1.229 - 0.367\rho_{12} + 0.153\rho_{12}^2$ and $\rho_{12}^0 = R_{12}$) we obtain $\rho_{12} = -0.5601$ while the numerical solution through double integration yields -0.5562. The relative error is 0.7% thus showing that either approach is satisfactory. The transformation matrix A is then determined by Cholesky decomposition of the correlation matrix ρ^0 :

$$A = \begin{bmatrix} 1 & 0 \\ -0.5562 & 0.8311 \end{bmatrix} \quad (29)$$

The limit state functions in X-space and U-space are depicted in Figures 8 and 9. The updated SVM boundary in U-space is shown in Figure 10. MCS samples used for failure probability calculation are plotted in Figure 11. The final SVM is constructed with 105 samples. Results of SVM update and failure probability calculation are presented in Table 2. For information, Figure 12 depicts the MCS samples when correlation is included and Figure 13 when correlation is ignored. The convergence plot for this problem is provided in Figure 14.

2. Exponential-Weibull Distributions with multiple failure modes

This example consists of four failure modes. A sample is considered failed if any of the modes is active (i.e., it is a series system). The failure domain Ω_f is union of the following modes:

$$\begin{aligned} g_1(x_1, x_2) &= 3 - (x_1 - 2)^3 - 3(x_2 - 1) \leq 0 \\ g_2(x_1, x_2) &= 2.5 - x_1 + x_2 \leq 0 \\ g_3(x_1, x_2) &= x_1^2 - 5x_1 + x_2 + 6.05 \leq 0 \\ g_4(x_1, x_2) &= 2x_1^2 - x_2 + 3 \leq 0 \end{aligned} \quad (30)$$

Table 1. Results of 2D and 3D problems.

| Results | Exp-Exp | Exp-Wbl | Norm-Exp-Wbl |
|--|----------------|----------------|------------------|
| $P_f^{SVM\dagger} (\times 10^{-2})$ | 5.587 | 5.024 | 0.2754 |
| $P_f^{Ref\dagger} (\times 10^{-2})$ | 5.676 | 5.065 | 0.2778 |
| $\epsilon^{SVM} (\%)$ | 1.566 | 0.567 | 0.864 |
| 95% CI of $P_f^{Ref} (\times 10^{-2})$ | [5.629, 5.722] | [5.021, 5.108] | [0.2673, 0.2883] |
| $P_f^{UC} (\times 10^{-2})$ | 9.851 | 3.718 | 1.074 |
| $\epsilon^{UC} (\%)$ | 73.568 | -26.597 | 286.645 |

* Assuming x_1 and x_2 are uncorrelated

† 1,000,000 samples are used in Monte Carlo simulation.

‡ Calculated by transforming back to original space

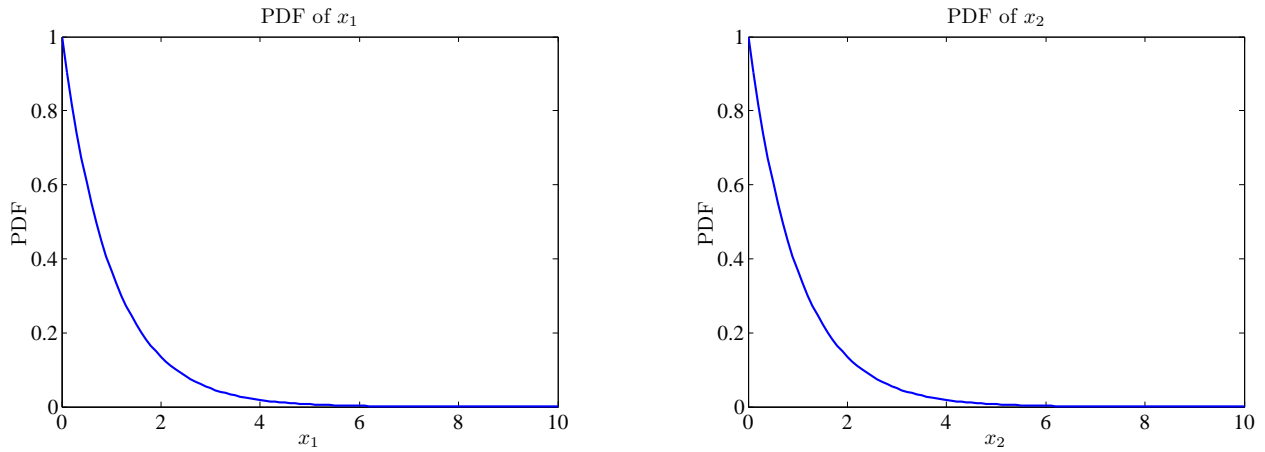


Figure 7. Distribution of correlated input variables

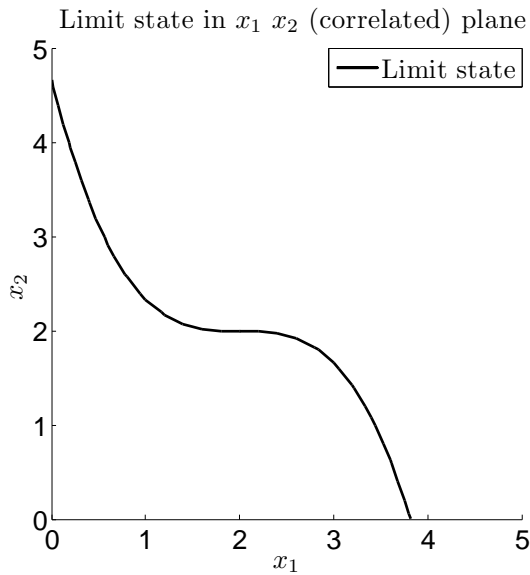


Figure 8. Limit state function in X-space

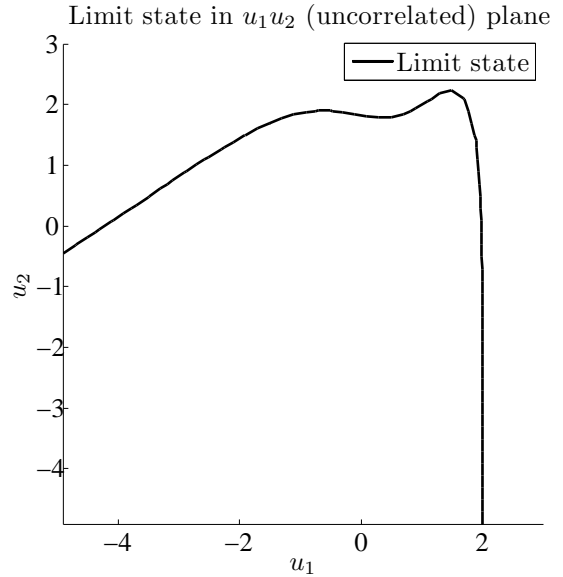


Figure 9. Limit state function in U-space

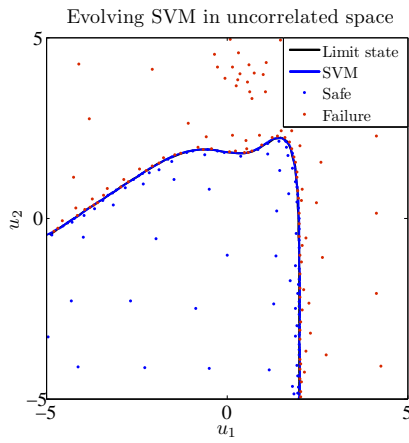


Figure 10. Updated SVM after adaptive sampling

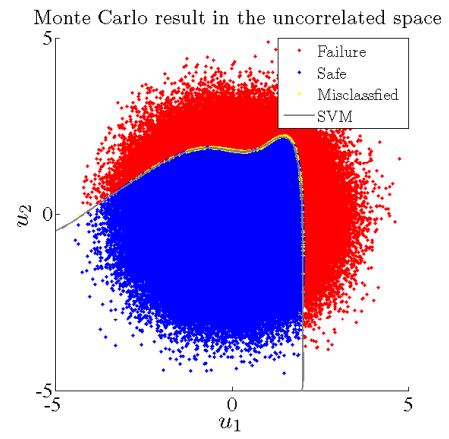


Figure 11. Monte Carlo samples in the uncorrelated standard normal space.

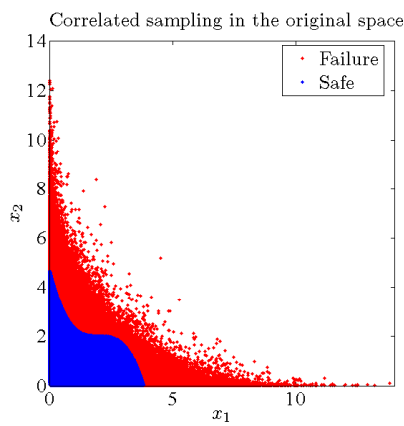


Figure 12. Transforming uncorrelated samples back to original space.

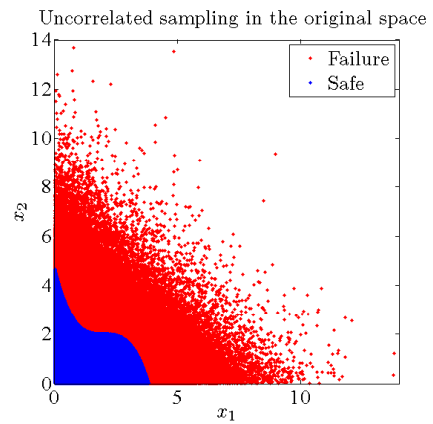


Figure 13. Direct sampling in original space ignoring correlation.

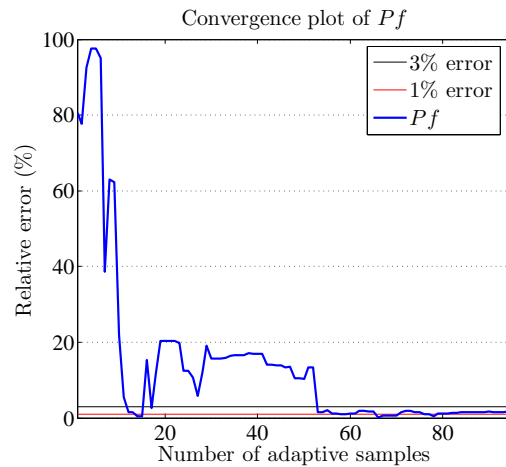


Figure 14. Convergence plot of probability of failure P_f .

x_1 is an exponential variable with mean of 1 and x_2 follows Weibull distribution with $\alpha = 1$ $\beta = 2$. The

distribution plots of the two variables are shown in Figure 15.

The composite limit state functions in X-space and U-space are shown in Figures 16 and 17. The updated SVM boundary in U-space is shown in Figure 18. MCS samples used for failure probability calculation are plotted in Figure 19. The updated SVM is constructed with 125 samples. Failure probability results are presented in Table 2.

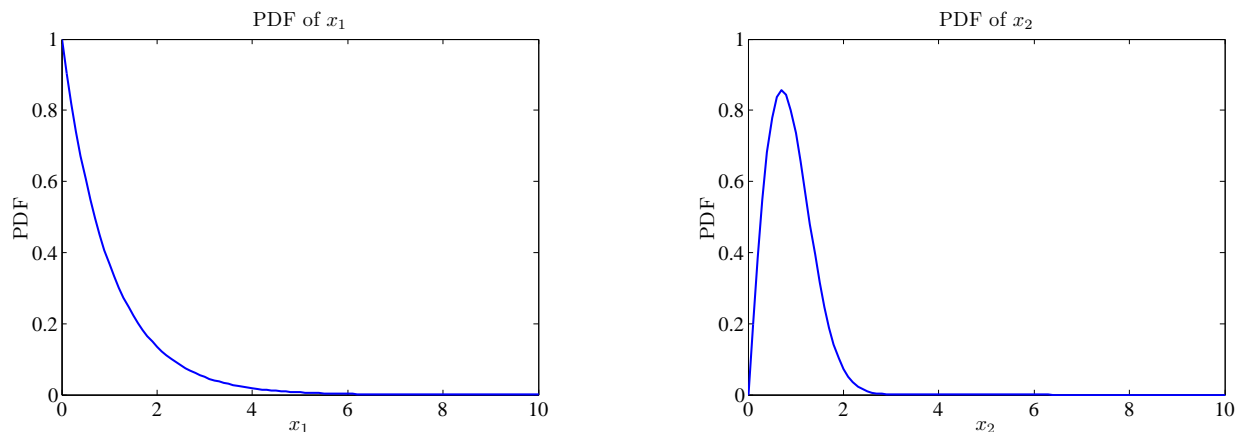


Figure 15. Distribution of correlated Exp-Wbl variables

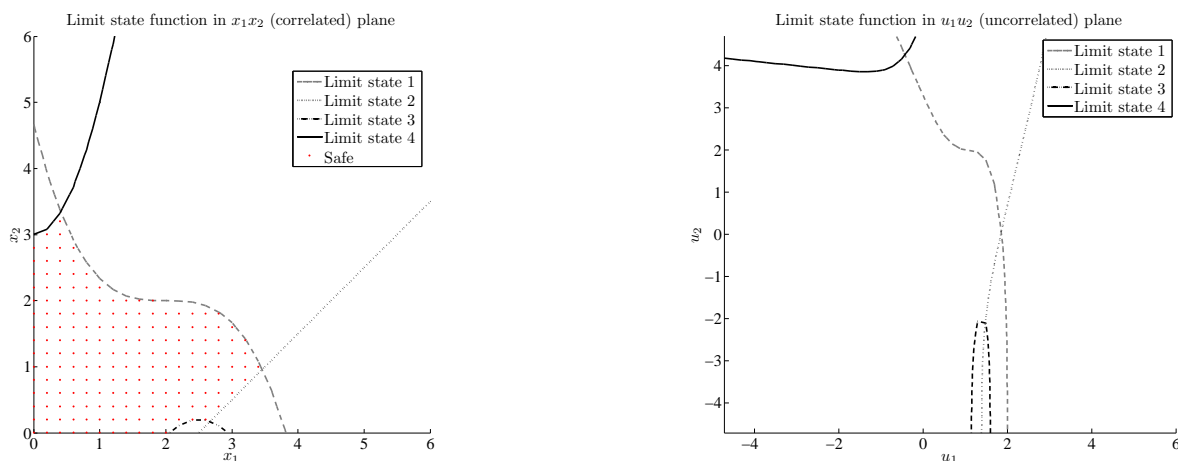


Figure 16. Multiple limit states in X-space

Figure 17. Multiple limit states in U-space

B. Three dimensional example

1. Normal-Exponential-Weibull Distribution

Extending the application to 3D case, consider the limit state function:

$$g(x_1, x_2, x_3) = -x_1^3 - 2x_2^3 - x_3^3 - 1.5x_1x_2x_3 + 9x_1^2 + 18x_2^2 + 18x_3^2 - 27x_1 - 54x_2 - 54x_3 + 136.5 = 0 \quad (31)$$

x_1 is a normal variable with 0 mean and standard deviation of 1. x_2 is an exponential variable with mean of 1. x_3 follows a Weibull distribution with $\alpha = 1.5$ $\beta = 2.5$. The distribution plots of the two variables are shown in Figure 21. The limit state functions in X-space and U-space are shown in Figures 22 and 23. The updated SVM boundary in U-space is shown in Figure 24. MCS samples used for failure probability calculation are plotted in Figure 25. The initial SVM is built with 50 CVT^{16,17} samples. At convergence,

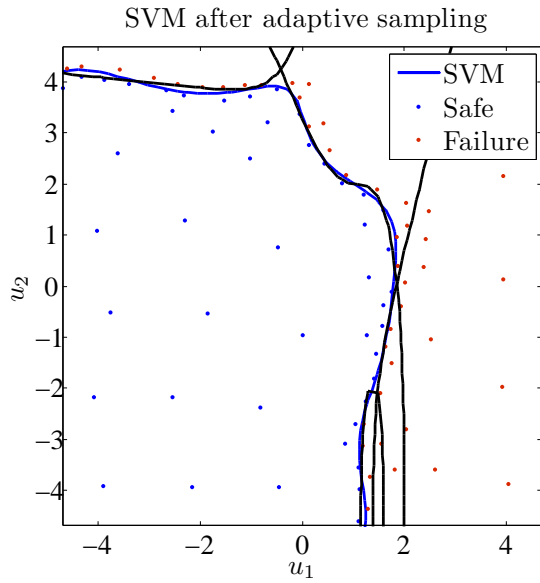


Figure 18. Updated SVM after adaptive sampling

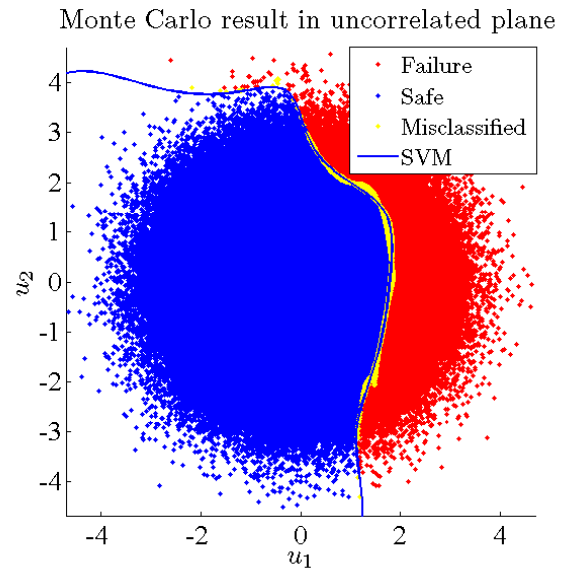


Figure 19. Monte Carlo samples classification based on updated SVM

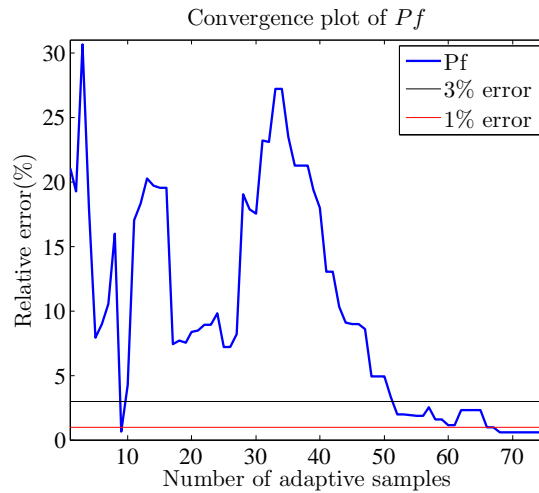


Figure 20. Convergence plot of probability of failure P_f .

the final SVM is constructed with 179 samples in total. Results of SVM update and failure probability calculation are presented in Table 1.

C. Two dimensional example with Probabilistic SVM (PSVM)

We consider the same two-dimensional problem with exponential-exponential distributions as presented in Section 1. The method based on PSVMs (see Section A) is used to provide a relatively conservative probability of failure with less number of function evaluations compared to the corresponding deterministic SVM. The convergence of probabilities of failure using the deterministic SVM and the PSVM are plotted in Figure 27. After adding 53 adaptive samples, it is observed that the probability of failure based on PSVM will always be larger than the reference probability calculated by direct transformation. On the contrary, the deterministic SVM can underestimate the probability of failure even with many more sample evaluations.

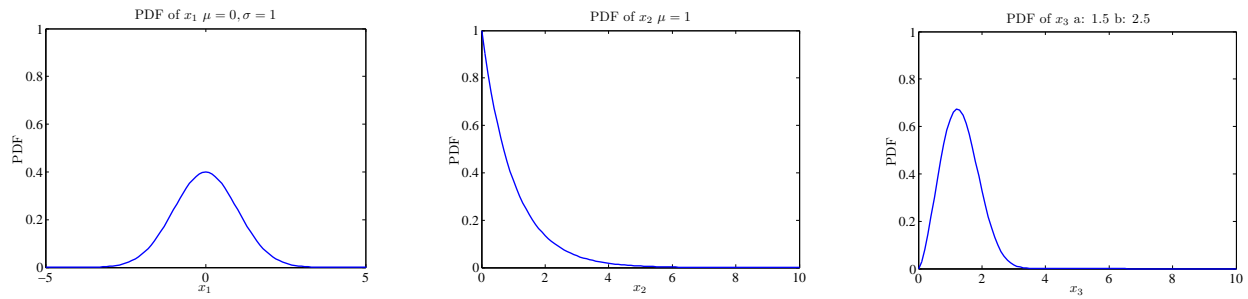


Figure 21. Distributions of correlated input variables for three dimensional problem

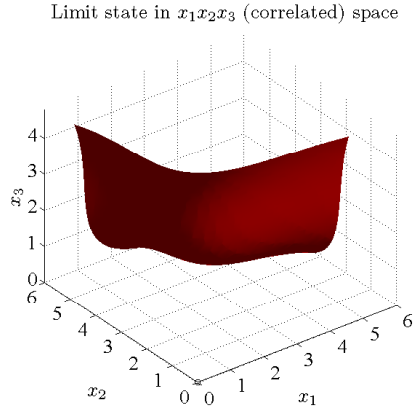


Figure 22. Limit state function in X-space

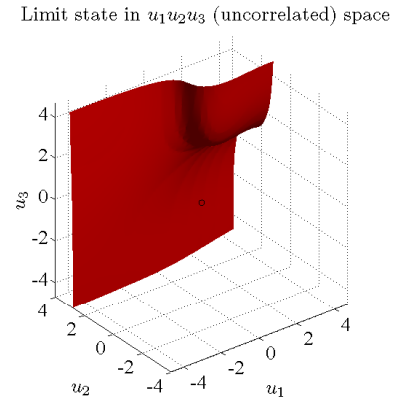


Figure 23. Limit state function in U-space

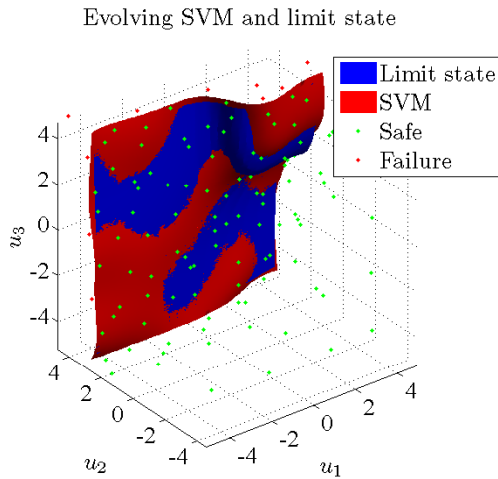


Figure 24. Updated SVM after adaptive sampling for the three dimensional problem

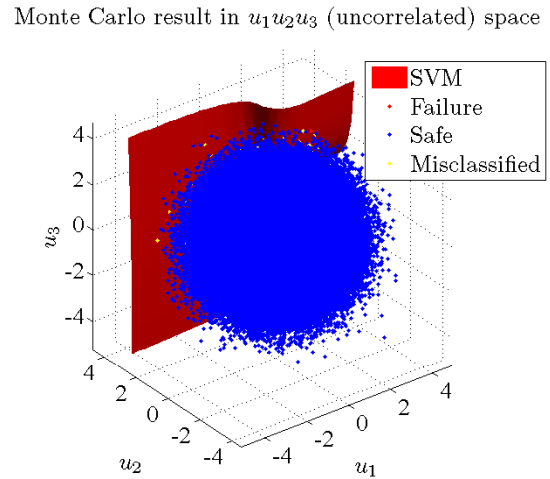


Figure 25. Monte Carlo samples classification based on updated SVM

However these observations are only valid for this example and cannot be generalized. More generally, the PSVM approach will require less function evaluations because it will always be, by construction, more conservative than the estimate based on the deterministic SVM. For example, if a less tight convergence criterion of 1% (instead of 0.1%) relative change of probability is used, the number of samples needed is 34. The deterministic SVM will give a lower probability of failure with an error of -16.366% , while the probability calculated based on PSVM will give an error of -0.640% (see Table 2).

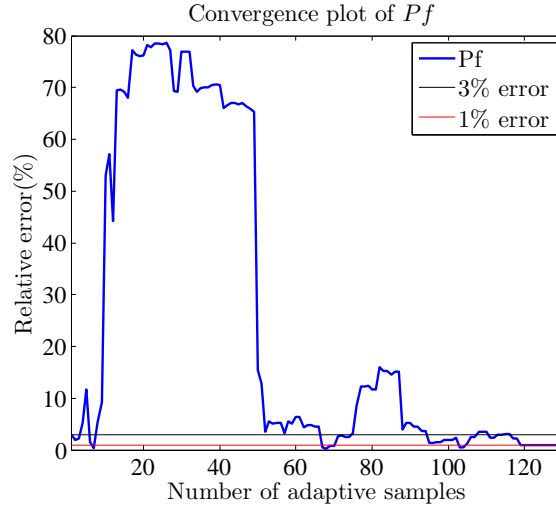


Figure 26. Convergence plot of probability of failure P_f for the three dimensional problem.

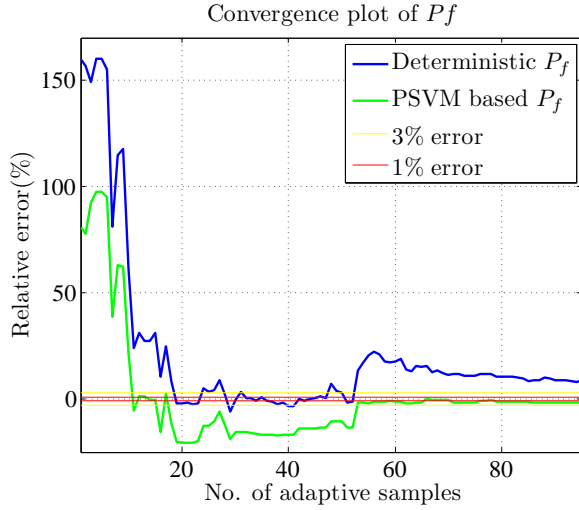


Figure 27. Convergence plot of deterministic P_f and P_f based on PSVM.

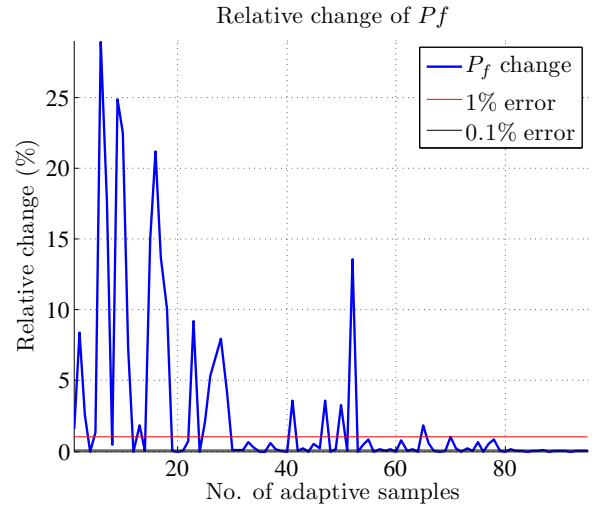


Figure 28. Relative change of P_f for stopping criteria

Table 2. Results of 2D Exp-Exp using PSVM.

| Results | 0.1% criteria | 1% criteria |
|---------------------------------------|---------------|-------------|
| No. of adaptive samples | 95 | 34 |
| $P_f^{PSVM} (\times 10^{-2})$ | 5.583 | 4.747 |
| $P_f^{PSVM} (\times 10^{-2})$ | 6.180 | 5.639 |
| $P_f^{Ref \ddagger} (\times 10^{-2})$ | 5.676 | 5.676 |
| $\epsilon^{PSVM} (\%)$ | -1.633 | -16.366 |
| $\epsilon^{PSVM} (\%)$ | 8.892 | -0.640 |

\ddagger Calculated by transforming back to original space

IX. Conclusion

This work in progress introduces an approach to evaluate probabilities of failure when the random variables are correlated. The technique is based on the explicit construction of the limit state function in the

uncorrelated standard normal space after Nataf transformation. The construction of an accurate limit state function is based on SVM explicit design space decomposition with adaptive sampling. Results with various random distributions and multiple failure modes are presented.

The next steps of this research will involved higher dimensional problems and reliability-based design optimization.

X. Acknowledgments

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