Reliability assessment using probabilistic support vector machines

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Abstract: This paper presents a methodology to calculate probabilities of failure using Probabilistic Support Vector Machines (PSVMs). Support Vector Machines (SVMs) have recently gained attention for reliability assessment because of several inherent advantages. Specifically, SVMs allow one to construct explicitly the boundary of a failure domain. In addition, they provide a technical solution for problems with discontinuities, binary responses, and multiple failure modes. However, the basic SVM boundary might be inaccurate; therefore leading to erroneous probability of failure estimates. This paper proposes to account for the inaccuracies of the SVM boundary in the calculation of the Monte Carlo-based probability of failure. This is achieved using a PSVM which provides the probability of misclassification of Monte Carlo samples. The probability of failure estimate is based on a new sigmoid-based PSVM model along with the identification of a region where the probability of misclassification is large. The PSVM-based probabilities of failure are, by construction, always more conservative than the deterministic SVM-based probability estimates.

Keywords: probabilistic support vector machines; probability of misclassification; probability of failure; discontinuous and binary behaviour; multiple failure modes.

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1 Introduction

The need for reliable, increasingly complex systems has led to a growing emphasis on efficient methods for design under uncertainty. As a result, reliability assessment techniques have been steadily improving. However, it is well understood that the calculation of a failure probability is hampered by hurdles such as large computational or experimental costs, discontinuous and binary system behaviours, and multiple failure modes (Missoum et al., 2007, Layman et al., 2010; Arenbeck et al., 2007).

Among reliability assessment methods, moment-based methods have been widely used. These include first- and second-order reliability methods (Haldar and Mahadevan, 2000), advanced mean value method (Youn et al., 2005), and two-Point Adaptive Non-linear Approximation (TANA) (Wang and Grandhi, 1995). These methods can, however, produce significant errors if the limit-state functions are highly non-linear (Bichon et al., 2007). In the last two decades, metamodelling and response surface techniques have gained prominence for the calculation of failure probabilities (Mourelatos et al., 2006; Simpson et al., 2008). In this approach, an approximation of the system responses is constructed based on a design of experiments. Adaptive sampling schemes to refine the approximations have also been developed (Bichon et al., 2007; Wang et al., 2005; Huang et al., 2006). While these methods can efficiently handle non-linear limit-state functions, they are limited by several other difficulties. For instance, the presence of discontinuous (Missoumet al., 2007; Basudhar et al., 2008; Basudhar and Missoum, 2009a; Basudhar and Missoum, 2009b) and binary (pass or fail) (Layman et al., 2010) responses poses difficulties for these approximation methods. In addition, the presence of multiple failure modes (Basudhar and Missoum, 2009a; Basudhar and Missoum, 2009b; Arenbeck et al., 2007) also presents a challenge.

Recently, an alternative sampling-based method was developed by the authors to overcome these difficulties. The approach, referred to as the 'Explicit Design Space Decomposition (EDSD)' (Basudhar et al., 2008), classifies the responses as 'safe' or 'failed'. Because there is no response approximation, the EDSD method can be used irrespective of the presence of discontinuous or binary behaviour (Basudhar et al., 2008; Layman et al., 2010). A machine learning technique referred to as Support Vector Machines (SVMs) (Vapnik, 1998; Gunn, 1998) is used to construct the explicit boundary that separates the safe and failed samples. EDSD is also useful for multiple failure modes because a single SVM can be used to represent the boundary of the failure domain (Arenbeck et al., 2007). Adaptive sampling schemes have been developed to construct accurate SVM approximations using moderate number of function evaluations (Basudhar and Missoum, 2008; Basudhar and Missoum, 2010). Once the approximate boundary of the failure region has been built, failure probabilities can be efficiently assessed using Monte Carlo Simulations (MCS) (Basudhar et al., 2008).

Although the recently developed methods have provided a welcome flexibility in reliability assessment, local or global inaccuracies in the SVM-based approximation of the limit-state function might lead to substantial errors in the failure probability. For this reason, this paper proposes to quantify the accuracy of the SVM classification model and propagate this information to the calculation of the probability of failure. More specifically, the probability of failure is calculated by weighing the traditional indicator function of the Monte Carlo samples by a probability of misclassification. This

probability of misclassification is only calculated within a region of the safe region based on the spatial distribution of the training samples. Therefore, the modified, MCS-based, measure of probability of failure is constructed so that it is always more conservative compared to the probability of failure based on a deterministic SVM, thus mitigating the consequences of an inaccurate SVM approximation.

The probability of misclassification is obtained using a Probabilistic Support Vector Machine (PSVM). This paper proposes necessary enhancements to existing PSVM models (Vapnik, 1998; Wahba, 1999; Platt, 1999). The proposed new PSVM model, referred to as distance-based PSVM (DPSVM), accounts for the spatial distribution of data points.

The proposed method is tested on two analytical examples. The main objective of the examples is to compare the probability of failure estimates to the actual probability but also to the case where the SVM is deterministic. This study is carried out for various designs of experiment sizes.

The organisation of the paper is as follows. A review of the basic SVM-based reliability assessment methodology is presented in Section 2. Section 3 presents the proposed PSVM-based reliability assessment method. The formulation of the failure probability accounting for the probability of misclassification by the SVM is explained. Section 4 provides an overview of the current PSVM models and their limitations. In Section 5, the modified DPSVM model proposed to overcome these limitations is presented. An error measure to compare the PSVM models is provided in Section 6. Finally, analytical test examples are presented in Section 7 to show the efficacy of the proposed method, followed by a discussion in Section 8.

2   Reliability assessment using support vector machines

This section provides a review of the previously developed method for reliability assessment using SVMs (Vapnik, 1998; Gunn, 1998). SVM is a machine learning technique that is widely used for the classification of data. In the context of reliability assessment, any given configuration or sample in the space is either safe or failed. Therefore, the definition of the failure domain can easily be treated as a binary classification problem (Basudhar et al., 2008; Hurtado, 2004). An SVM is used to define the explicit boundary of the failure region that separates the safe and failed samples:

$$s(\mathbf{x}) = b + \sum_{i=1}^{N} \lambda_i y_i K(\mathbf{x}_i, \mathbf{x}) = 0 \tag{1}$$

Here, $\mathbf{x}_i$ is the i-th training sample, λ_i is the corresponding Lagrange multiplier, N is the number of training samples, K is a kernel function, y_i is the class label corresponding to $\mathbf{x}_i$ that can take values $+1$ and -1, and b is the bias. The optimal SVM boundary is constructed by maximising the 'margin' between $s(\mathbf{x}) = +1$ and $s(\mathbf{x}) = -1$ in a 'feature space' (Vapnik, 1998), where the boundary is linear. If a so-called hard classifier is used, none of the training samples can lie within the margin. Once the SVM is constructed, the class of any point $\mathbf{x}$ is predicted based on the sign of $s(\mathbf{x})$. If $s(\mathbf{x}) > 0$ (resp. $s(\mathbf{x}) < 0$), it is classified as $+1$ (resp. -1).

The main steps of the reliability assessment procedure using an SVM are as follows.

- Sampling of the space using a Design of Experiments (DOE) (Montgomery, 2005; Beachkofski and Grandhi, 2002; Romero et al., 2006).

- Classification of the samples into two classes – safe and failed. The convention is to denote the safe samples as +1 and the failed ones as –1.

- Construction of the SVM limit-state boundary using equation (1).

- Calculation of the probability of failure using MCS (Melchers, 1999) based on the SVM boundary. The region $s(\mathbf{x}) \leq 0$ is considered as the failure region. The probability of failure is

$$P_f = \frac{N_f}{N_{MCS}} = \frac{1}{N_{MCS}} \sum_{i=1}^{N_{MCS}} I_g(\mathbf{x}_i), \tag{2}$$

where N_f is the number of MCS samples classified as failure by the SVM classifier [equation (1)] and N_{MCS} is the total number of MCS samples. $I_g(\mathbf{x})$ is an indicator function given as:

$$I_g(\mathbf{x}) = \begin{cases} 1 & s(\mathbf{x}) \leq 0 \\ 0 & s(\mathbf{x}) > 0 \end{cases} \tag{3}$$

An example of failure probability calculation using SVM is shown in Figure 1.

Figure 1 Calculation of the failure probability using an SVM (left). Misclassification of the MCS samples by the SVM (right) (see online version for colours)
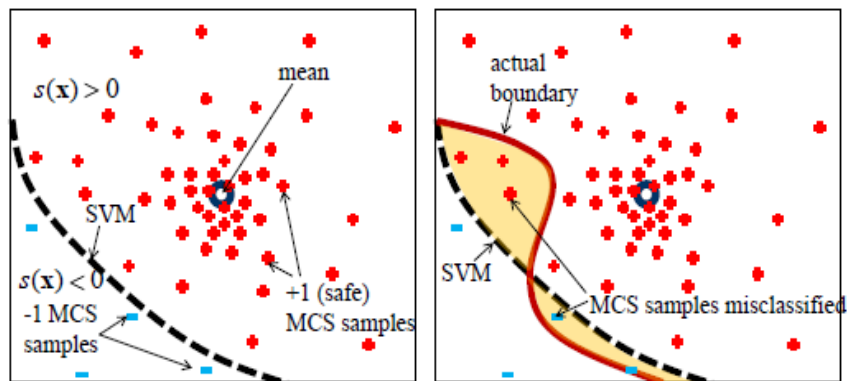
The calculation of probability of failure using the SVM-based method has several advantages. It is applicable to a wider range of problems compared to the current methods. These include problems with discontinuous and binary responses (Basudhar et al., 2008; Layman et al., 2010), and multiple failure modes (Arenbeck et al., 2007). Treating the reliability assessment process as a classification problem allows one to easily handle these difficulties.

The construction of the SVM limit-state function is based on a DOE. Therefore, there is, in general, an error associated with the approximation of the boundary. This can result in an inaccurate probability of failure (Figure 1). The following section introduces the use of PSVMs to estimate the probability of failure.

3   Reliability assessment using Probabilistic Support Vector Machines (PSVMs)

This section presents a method for calculating the probability of failure that accounts for the error associated with the SVM approximation. To do so, the probability of failure, evaluated using MCS, is calculated based on the probability of misclassification of the Monte Carlo samples in addition to the probability density functions of the variables. The probability of misclassification is calculated using a PSVM (Platt, 1999; Vapnik, 1998). A deterministic SVM (Section 2) only provides a binary classification. However, a PSVM provides the probability that a particular configuration (or sample) will belong to a specific class (+1 or −1). This probability, which is the conditional probability of belonging to the +1 (resp. −1) class is denoted as $P(+1|\mathbf{x})$ (resp. $P(-1|\mathbf{x})$).

It is noticed in equation (3) that the indicator function $I_g(\mathbf{x})$ can be interpreted as the probability of being in the failure class based on a deterministic SVM boundary. Therefore, the probability of being in the failure class −1 for any Monte Carlo sample is either 0 or 1. It is equal to 0 for a sample lying in the safe or +1 class and equal to 1 for a sample lying in the failure or −1 class. The use of PSVM allows one to replace the binary indicator function by $P(-1|\mathbf{x}_i)$, thus leading to:

$$P_f^{PSVM} = \frac{1}{N_{MCS}}\left(\sum_{i=1}^{N_{MCS}} P(-1|\mathbf{x}_i) \right) \tag{4}$$

A relatively conservative measure of the probability of failure is obtained if the probability of misclassification is considered only for the Monte Carlo samples belonging to the safe class, that is, for $s(\mathbf{x}) > 0$:

$$P_f^{PSVM} = \frac{1}{N_{MCS}}\left(\sum_{i=1}^{N_{MCS}} \psi(-1|\mathbf{x}_i) \right),$$

$$\psi(-1|\mathbf{x}) = \begin{cases} 1 & s(\mathbf{x}) \le 0 \\ P(-1|\mathbf{x}) & s(\mathbf{x}) > 0 \end{cases} \tag{5}$$

In equation (5), the probability of misclassification is considered only for the samples lying in the safe domain based on the SVM $(s(\mathbf{x}) > 0)$. Therefore, it would naturally provide a probability of failure that is greater than the one using the deterministic SVM [equation (2)]. However, the failure probability estimate using equation (5) may be over-conservative. Therefore, instead of considering a non-zero $P(-1|\mathbf{x})$ for the entire safe domain, it is reasonable to consider it only in the regions with high probability of misclassification. Such regions can be identified as the ones 'lacking data' in the vicinity of the SVM boundary. Therefore, the region Ω_{misc} for considering the probability of misclassification is a sub-region of the $s(\mathbf{x}) > 0$ regions. It is defined based on the

distances to the closest +1 and −1 samples and the SVM margin, which does not contain any samples. Outside this region, the classification provided by the deterministic SVM is trusted (i.e. $P(-1 \,|\, \mathbf{x})$ is either 1 or 0). The region Ω_{misc} for considering the probability of misclassification by the SVM is

$$\Omega_{misc} = \Omega_{sd} \cap \Omega\big(|\,s(\mathbf{x})\,| < 1 \cup (d_+(\mathbf{x}) - d_-(\mathbf{x}) \geq 0)\big), \tag{6}$$

where Ω_{sd} is the safe domain based on the deterministic SVM, and $d_+(\mathbf{x})$ and $d_-(\mathbf{x})$ are the distances of $\mathbf{x}$ to the closest +1 and −1 training samples. Ω_{misc} consists of two kinds of regions in the +1 class. One is the SVM margin in the safe class and another is the region with $d_+(\mathbf{x}) \geq d_-(\mathbf{x})$ (Figure 2). The probability of failure is given as:

$$P_f^{PSVM} = \frac{1}{N_{MCS}}\left(\sum_{i=1}^{N_{MCS}} \gamma(-1\,|\,\mathbf{x_i})\right),$$

$$\gamma(-1\,|\,\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \Omega_f \\ 0 & \mathbf{x} \in \Omega_{sd} - \Omega_{misc} \\ P(-1\,|\,\mathbf{x}) & \mathbf{x} \in \Omega_{misc} \end{cases} \tag{7}$$

Figure 2 Definition of the region Ω_{misc} for considering the probability of misclassification. Ω_{misc} is the union of the two shaded regions in the left and the right figures (see online version for colours)
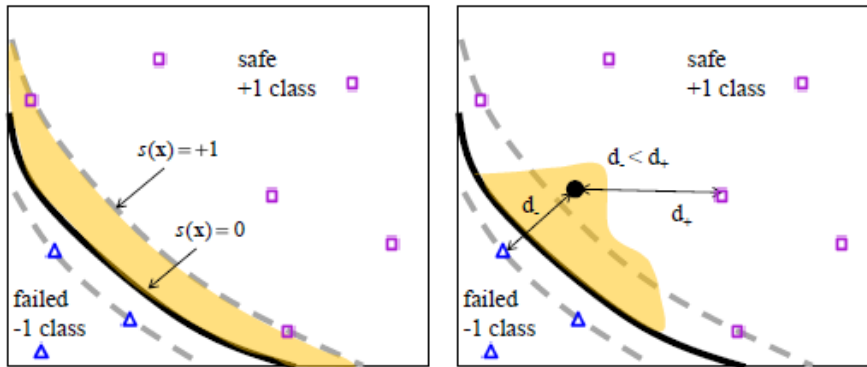
The calculation of the failure probability using equation (7) requires the calculation of $P(-1\,|\,\mathbf{x})$ in the region Ω_{misc}. The details of the PSVM models for calculating $P(+1\,|\,\mathbf{x})$ or $P(-1\,|\,\mathbf{x})$ are presented in the following sections.

4 Traditional sigmoid-based PSVM model

Probabilistic support vector machines are used to map the SVM values to the probability of belonging to a specific class. One of the commonly used PSVM models proposed by Platt (1999) is based on the fitting of a sigmoid function. Some other models have also been proposed by Wahba (1999) and Vapnik (1998). The sigmoid PSVM model is presented in the following section.

4.1　Sigmoid model for Probabilistic Support Vector Machines (PSVMs)

The sigmoid PSVM model proposed by Platt is presented. In this model, the conditional probability $P(+1|\mathbf{x})$ is represented as a function of two parameters A and B:

$$P(+1|\mathbf{x}) = \frac{1}{1 + e^{As(\mathbf{x})+B}} \tag{8}$$

The parameters A $(A < 0)$ and B of the sigmoid function are found using maximum likelihood. The conditional probability of correctly classifying the sample $\mathbf{x}_i$ is $P(+1|\mathbf{x}_i)$ if it belongs to the $+1$ class and is equal to $1 - P(+1|\mathbf{x}_i)$ if it belongs to the -1 class. The information available to train the PSVM is the class of the training samples. According to equation (8), the likelihood that all the training samples are classified correctly is a function of A and B:

$$L(A,B) = \prod_{i=1}^{N_p} P(+1|\mathbf{x}_i) \prod_{i=1}^{N_n} \left(1 - P(+1|\mathbf{x}_i)\right), \tag{9}$$

where N_p is the number of $+1$ sample and N_n is the number of -1 sample. A good PSVM model should provide a high likelihood of correctly classifying all the samples. Therefore, the parameters A and B are found by maximising $L(A, B)$. In the work of Platt (1999), the log of the likelihood function was used as:

$$\min_{A,B} \quad -\sum_{i}^{N} t_i log\left(P(+1|\mathbf{x}_i)\right) + \left(1 - t_i\right) log\left(1 - P(+1|\mathbf{x}_i)\right), \tag{10}$$

where N is the number of training samples and t_i is given as:

$$t_i = \frac{y_i + 1}{2}, \tag{11}$$

where y_i is the class labels. Thus, $t = 1$ for the $+1$ samples and $t = 0$ for the samples belonging to the -1 class.

4.2　Limitations of the basic sigmoid model

One of the limitations of the model mentioned in Section 4.1 is that it depends only on the SVM values and not on the spatial distribution of the samples. As a result, if the classes of the evaluated samples are considered deterministic, it does not satisfy one of the conditions that requires $P(+1|\mathbf{x})$ to be either 0 or 1 at these samples. Instead, it provides a non-zero probability of misclassification (e.g. $P(+1|\mathbf{x}) > 0$ for $s(\mathbf{x}) < 0$) even for the samples that have already been evaluated. An example of the probability of misclassification P_{misc} using Platt's sigmoid model is shown in Figure 3. It is seen that the probability of misclassification is high even for regions that are far from the boundary and consist of already evaluated samples.

Figure 3 Map of the probabilities of misclassification using Platt's sigmoid model (see online version for colours)
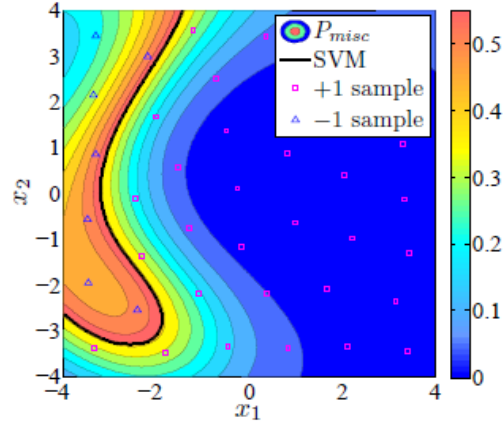
5 Improved Distance-based Probabilistic Support Vector Machines (DPSVMs) using a modified sigmoid model

The biggest limitation of the basic sigmoid PSVM stems from neglecting the spatial distribution of the evaluated samples. To overcome this issue, a modified sigmoid model is presented in this section.

The proposed model depends not only on the SVM values, but also on the distances to the evaluated samples used to train the SVM. Because the proposed model depends on the spatial distribution of the samples, it is also referred to as Distance-based Probabilistic Support Vector Machine (DPSVM). It is assumed that in this model the class of any evaluated sample is deterministic. The modified sigmoid model is defined as:

$$P(+1\,|\,\mathbf{x}) = \frac{1}{1+e^{As(\mathbf{x})+B(\frac{d_-}{d_++\tau}-\frac{d_+}{d_-+\tau})}} \qquad A < \frac{-3}{\min(s_{max},-s_{min})}, B < 0, \qquad (12)$$

where d_- and d_+ are the distances to the closest -1 and $+1$ samples, respectively. s_{min} and s_{max} are the minimum and maximum SVM values among the training samples. τ is a small quantity (set equal to 10^{-10} in this work) added in order to avoid numerical issues at the evaluated training sample.

The proposed model satisfies the following boundary conditions:

$P(+1\,|\,\mathbf{x}) \to 1$ if $s(\mathbf{x}) \to \infty$ or $d_+ \to 0$

$P(+1\,|\,\mathbf{x}) \to 0$ if $s(\mathbf{x}) \to -\infty$ or $d_- \to 0$

$P(+1\,|\,\mathbf{x}) \to 0.5$ if $s(\mathbf{x}) \to 0$ and $d_- \to d_+$

The upper bound on A ensures that $P(+1|\mathbf{x})$ does not have a strong dependence on the distances away from the boundary. That is, the values of $P(+1|\mathbf{x})$ are close to 0 or 1 far from the boundary irrespective of the influence of the distances. More specifically, for $B = 0$, the upper bound ensures $P(+1|\mathbf{x}) > 0.95$ at the point of maximum SVM value s_{max} and $P(+1|\mathbf{x}) < 0.05$ at the point of minimum SVM value s_{min}. The proof of the former is given below by setting $B = 0$:

$$\frac{1}{1+e^{As_{max}}} > 0.95$$

$$\Rightarrow A < \frac{\ln\left(\frac{0.05}{0.95}\right)}{s_{max}} = \frac{-2.94}{s_{max}} \approx \frac{-3}{s_{max}} \tag{13}$$

The strict inequality bound on B ($B < 0$) ensures that $P(+1|\mathbf{x}) \to 1$ at the +1 sample and $P(+1|\mathbf{x}) \to 0$ at the −1 sample. The proof for the two cases is given below. Setting $\tau = 0$ for the proof, we have

For the +1 samples, $d_+ \to 0$:

$$P(+1|\mathbf{x}) = \lim_{d_+ \to 0} \frac{1}{1+e^{As(\mathbf{x})+B(\frac{d_-}{d_+})}} \approx \frac{1}{1+e^{As(\mathbf{x})-\infty}} \approx \frac{1}{1+e^{-\infty}} \approx 1 \tag{14}$$

For the −1 sample, $d_- \to 0$:

$$P(+1|\mathbf{x}) = \lim_{d_- \to 0} \frac{1}{1+e^{As(\mathbf{x})+B(-\frac{d_+}{d_-})}} \approx \frac{1}{1+e^{As(\mathbf{x})+\infty}} \approx \frac{1}{1+e^{+\infty}} \approx 0 \tag{15}$$

The training process for the DPSVM is as follows:

- The values of d_+ for a +1 sample and that of d_- for a −1 sample are zero. However, during the training process, these are assigned as the distances to the closest +1 and −1 samples other than the sample under consideration.

- The values of $s(\mathbf{x})$, d_- and d_+ at training samples are used to calculate the likelihood function, which is then maximised to find A and B. The optimisation is solved using a genetic algorithm (GA).

Because the proposed DPSVM model accounts for both the SVM values and the spatial distribution of the evaluated samples, it overcomes the limitations of the basic SVM model mentioned in Section 4.2. A conceptual graphical comparison of the proposed model with Platt's sigmoid model is provided in Figure 4. A map of the probabilities of misclassification is shown in Figure 5.

Figure 4   Comparison of the two PSVM models (see online version for colours)
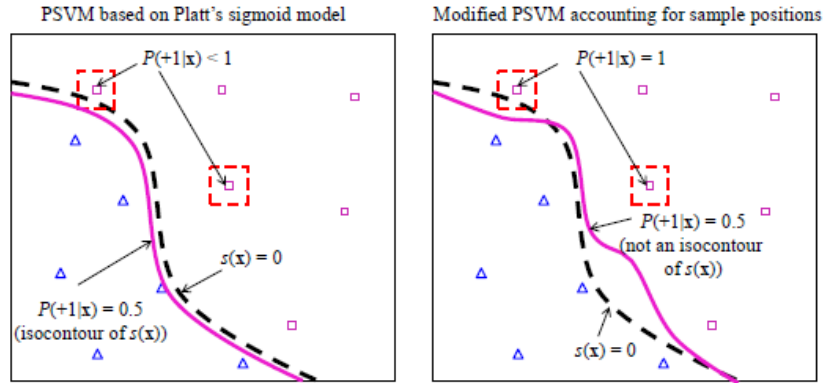
Figure 5   Map of the probabilities of misclassification using the modified distance-based sigmoid model. A comparison with the probabilities in Figure 2 shows major differences (see online version for colours)
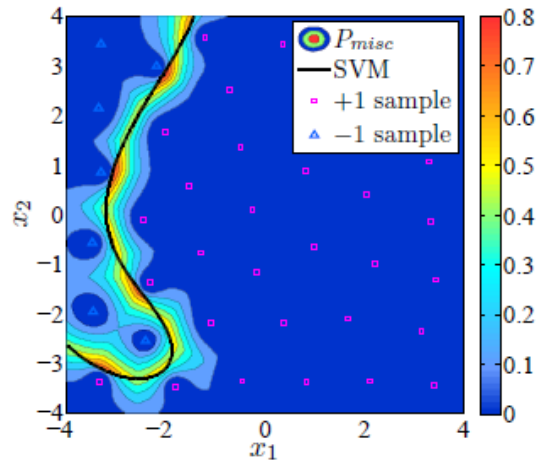
6   Error quantification of the PSVM model

In order to compare the proposed DPSVM model with Platt's PSVM model, a measure to quantify the error for the models is presented in this section. In the case where the actual limit-state function is known, $P(+1 \,|\, \mathbf{x})$ is known for any point and is equal to 0 or 1. Therefore, the error of the PSVM model can be calculated at any point. A large number of test points from a uniform grid are used for this purpose. Because the actual class of all the test points is known, the probability of misclassification for the *i*-th point is

$$P_{misc}\left(\mathbf{x}_i\right) = \begin{cases} 1 - P\left(+1 \,|\, \mathbf{x}_i\right) & s(\mathbf{x}_i) > 0 \\ P\left(+1 \,|\, \mathbf{x}_i\right) & s(\mathbf{x}_i) < 0 \end{cases}, \tag{16}$$

where $\mathbf{x}_i$ represents the *i*-th test point. A good PSVM model should provide a low probability of misclassification for the test points. The error E_{test} is defined as the mean probability of misclassification for all the test points:

$$E_{test} = \frac{1}{N_{test}} \sum_{i=1}^{N_{test}} P_{misc}(\mathbf{x}_i) \tag{17}$$

7   Examples

This section presents analytical test examples to demonstrate the proposed failure probability measure. The examples consist of two and three variables. For each example, the boundary of the failure region is approximated with a polynomial kernel SVM trained with a CVT DOE. The degree of the polynomial is selected as the lowest without training misclassification (Basudhar and Missoum, 2010). For each example, the ratio of the probability of failure obtained with the DPSVM model to the probability obtained with the deterministic SVM is provided. This ratio, referred to as the probability ratio (*PR*), provides a measure of the conservativeness of the DPSVM model compared to the deterministic SVM. The size of the DOE is varied to study its effect on the probability of failure. For comparing the probabilities of failure, all the variables are assumed to have truncated Gaussian distributions with zero mean and standard deviation equal to 1.0. The lower and upper bounds of all the variables are –4.0 and 4.0, respectively. The probabilities of failure are calculated using 10^6 MCS samples for all the examples. Example 1 is also used to compare the DPSVM model with the Platt's model. As explained in Section 6, a uniform grid is used to quantify the efficacy of the DPSVM model.

The following notations are used in this section:

- E_{test}^{Platt} : test point-based error for the Platt PSVM model;

- $E_{test}^{Modified}$: test point-based error for the DPSVM model;

- P_f^{Actual} : probability of failure calculated using the actual limit-state function;

- P_f^{SVM} : probability of failure calculated using the SVM limit-state function;

- $P_f^{Modified}$: PSVM-based probability of failure calculated using the DPSVM model;

- $\varepsilon_{P_f}^{SVM}$: relative difference between P_f^{SVM} and P_f^{actual} ;

- $\varepsilon_{P_f}^{Modified}$: relative difference between $P_f^{Modified}$ and P_f^{actual} ;

- *PR*: probability ratio $\dfrac{P_f^{Modified}}{P_f^{SVM}}$: ratio of failure probability obtained with DPSVM model to probability obtained with deterministic SVM.

7.1 Example 1 – two-dimensional problem

This example consisting of two variables $x_1, x_2 \in [-4, 4]$ has two failure modes. The failure region, for a parallel system, is defined as:

$$\Omega_f = \left(-8(x_1 - 2) + x_2^2 \leq 0\right) \cap \left(x_2 - \tan\left(\frac{\pi}{12}\right)(x_1 + 7) + 4 \leq 0\right) \tag{18}$$

The two modes and the resulting limit-state function are shown in Figure 6. With the SVM-based approach, both modes are represented by a single boundary.

Figure 6 Example 1: The left figure shows the limit-state functions for two failure modes. The right figure shows the net failure region (see online version for colours)
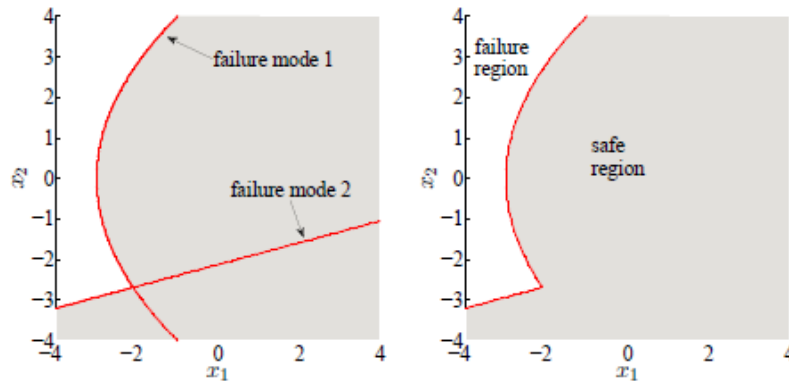
In order to study the effect of the DOE size, the SVM approximations are constructed using 40–100 CVT DOE samples with increments of 20. A comparison of the Platt's model and the proposed DPSVM model is provided in Section 7.1.1. Additionally, the probabilities of failure are provided in Section 7.1.2.

7.1.1 Comparison of the PSVM models

The two PSVM models are compared in this section based on the measure provided in Section 6. Sixteen hundred grid points are used to calculate the errors due to the two models. One example of the distribution of P_{misc} values in the space for 40 samples is shown in Figures 3 and 5. The errors E_{test}^{Platt} and $E_{test}^{Modified}$ are shown in Figure 7. The proposed modified PSVM model provides lower errors irrespective of the size of the DOE.

7.1.2 Comparison of the probabilities of failure

The probabilities of failure using varying sized DOEs are provided in Figure 8. The dashed-dotted green curve represents the probability of failure calculated using the deterministic SVMs. There is a significant variation in the failure probability depending on the DOE.

Figure 7   Example 1: Testing error for the PSVM models (see online version for colours)
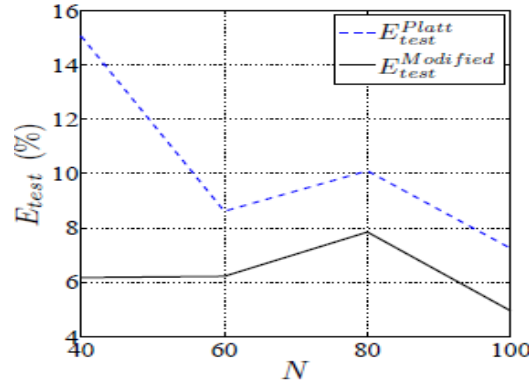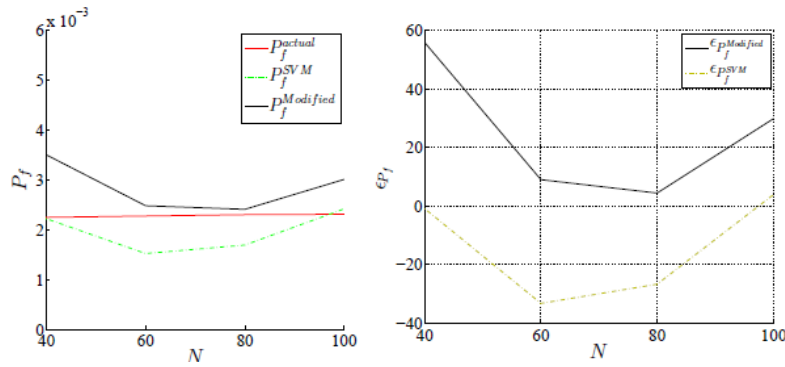

Figure 8   Example 1: Probabilities of failure and relative error as a function of the number of DOE samples. The probabilities of failure based on a deterministic SVM and the DPSVM model are presented (see online version for colours)


The DPSVM-based probability of failure and the relative differences with respect to P_f^{actual} are also shown in Figure 8 with the solid black curves in left and right plots. The relative differences between the deterministic SVM-based failure probability and the actual one are shown with the dashed olive green curve in the right hand side plot in Figure 8. It is seen from the figures that the deterministic SVM-based failure probability is less than the actual value (red) for several cases. The DPSVM-based failure probability is always, by construction, more conservative than the deterministic SVM case as demonstrated by the probability ratios depicted in Figure 9. Note that this ratio will tend, by construction of the DPSVM model, to 1 as the number of DOE samples tends to infinity.

7.2   *Example 2 – Three-dimensional problem*

A three variable example consisting of four disjoint regions in the space (Figure 10) is presented in this section. All the variables lie between –4 and 4. The failure region is given as:
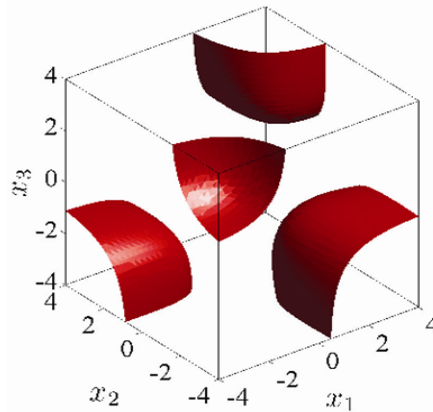
$$\Omega_f = (x_1 - 2)^2 + x_2^2 + (x_3 + 2)^2 - 3x_1 x_2 x_3 + 1 \leq 0 \tag{19}$$

Figure 9 Example 1: Probability ratio with respect to the deterministic SVM-based failure
probability
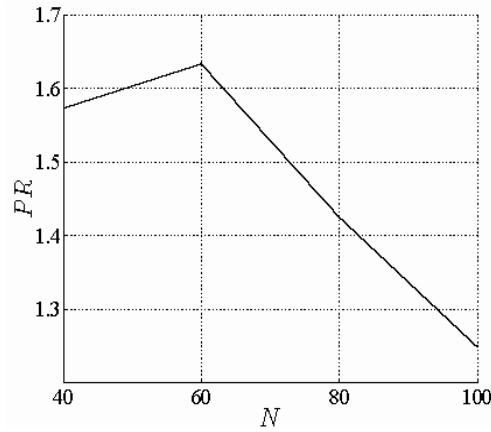
The SVM approximation of the limit-state function is constructed using 40–1000 samples
at an interval of 40.

Figure 10 Example 2: Actual limit-state function (see online version for colours)

Similar to Example 1, it is seen in Figure 11 that the probability of failure is calculated
using the deterministic SVM (dashed-dotted green) and PSVM shows significant
variation with respect to the size of the DOE. Also, using the deterministic SVM, the
failure probability is less than the actual value (red) in several cases. The DPSVM-based
failure probabilities are always more conservative than the deterministic SVM case as
demonstrated by the probability ratios depicted in Figure 12. As expected, the ratio
reduces with the number of samples because the confidence in the SVM increases. By
construction of the SVM, the ratio will tend to unity as the number of samples tends to
infinity.

Figure 11 Example 2: Probabilities of failure and relative error as a function of the number of DOE samples. The probabilities of failure based on a deterministic SVM and the DPSVM model are presented (see online version for colours)
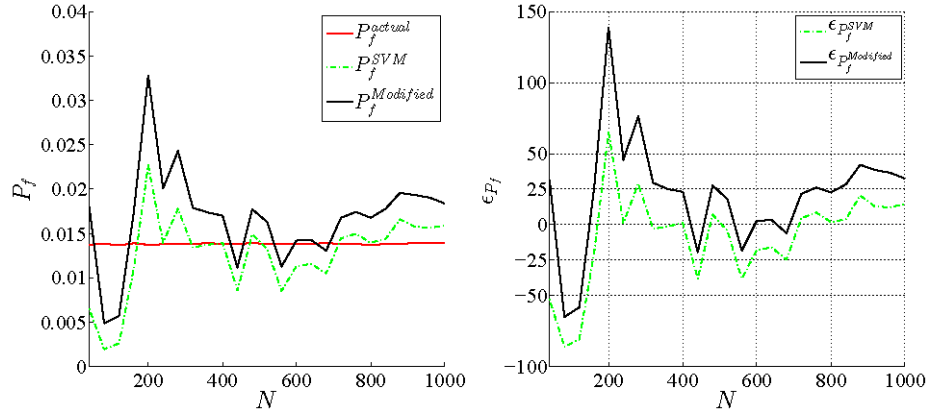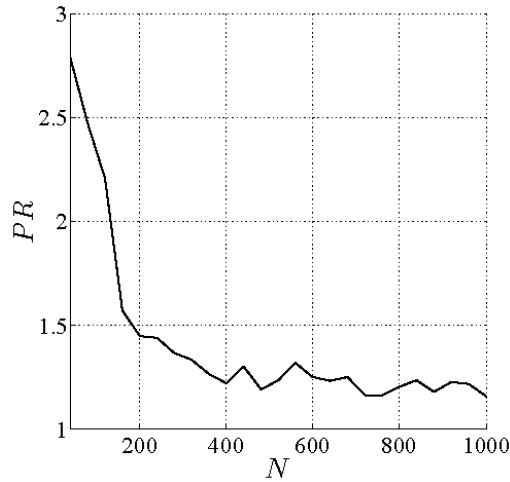

Figure 12 Example 2: Probability ratios with respect to the deterministic SVM-based failure probability


8   Discussion

This section presents a discussion on the results presented in Section 7.

- *Conservativeness of the PSVM-based probability estimate*: By construction, the DPSVM-based failure probability is calculated such that it is always more conservative than the one calculated with the deterministic SVM. Figures 9 and 12 show the ratio of the DPSVM-based failure probabilities to the probabilities using the deterministic SVMs. The ratio is always greater than 1. In fact, the probability ratios are higher when the sparsity is greater and reduces with the size of the DOE.

This indicates that the confidence in the SVM increases with the amount of data, as expected. In addition to considering the probability of misclassification by the SVM, it is also useful to consider the variance of the MCS. This is done in order to check if there is any overlap between the DPSVM and the deterministic failure probability prediction. For this purpose, 99% confidence intervals of the MCS failure probability estimate for Example 1 are depicted in Figure 13. They show that there is no overlap between the deterministic and the modified PSVM model.

- *DPSVM vs. Platt PSVM model*: In Example 1, the results of the comparison between the two PSVM models show a very clear trend. The comparison of the errors E_{test}^{Platt} and $E_{test}^{Modified}$ also shows lower errors. Both the errors reduce with the size of the DOE. However, the most important point is that, unlike the modified model, the Platt's model does not satisfy the condition of having probabilities equal to 0 or 1 at the evaluated samples. This latter point is to be satisfied if we consider deterministic computer experiments and do not account for errors associated with the model. If uncertainties on the model itself are to be considered, the Platt's model might be used.

- *Adaptive sampling*: In the studies performed in this paper, significant variations of the failure probabilities were observed with respect to the size of the CVT DOEs. Apart from using PSVMs for quantifying the probability of misclassification, another option to reduce the errors is to use adaptive sampling for the SVMs. Several adaptive sampling schemes for the update of SVMs have been developed in previous research (Basudhar and Missoum, 2008; Basudhar and Missoum, 2010). A new sampling scheme is also possible based on the proposed modified PSVM model. Because it provides the probability of misclassification, samples may be added in regions with high misclassification probability. This is, however, not the focus of this paper. The aim of this paper is to present a measure of the probability of failure based on PSVM. Even with adaptive sampling, the SVM may not always be accurate and it might be necessary to consider the probability of misclassification using PSVMs.

Figure 13 99% confidence interval of the MCS failure probability estimates for Example 1 (see online version for colours)
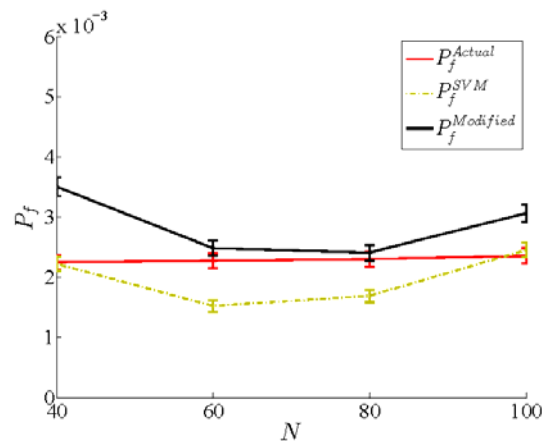
9   Conclusion

A method for reliability assessment using PSVMs was presented in this paper. The main idea is to include the probability of misclassification of Monte Carlo samples in the failure probability calculation. This probability of misclassification is calculated within a region in the safe domain defined using the spatial distribution of the samples. For this reason, the proposed failure probability measure is always, by construction, more conservative than the deterministic SVM which might present significant error. Apart from the failure probability measure, a modified PSVM model was also presented in this paper.

The next steps of this research will study higher dimensional examples. In addition, research will be conducted to use PSVM for adaptive sampling.

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