A Bayesian Approach for Identification using Support Vector Machines

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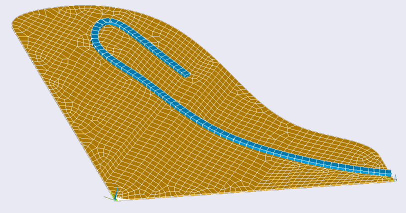
Outline

Introduction
Motivation

- Models are used to avoid experiments:
  - Need to be tuned
- Multiple needs for model updating:
  - Material properties
  - Multiple models
  - Crack detection
  - ...

Reality

Finite Element Model



Introduction
Challenges

However, model update process can be difficult if any of the following hurdles arise:

1. Expensive model
2. Numerous responses (multiple quantities to match)
   - *Fitting a joint PDF is difficult!*
3. Correlated responses

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Goal

This work aims at overcoming these issues.

Assuming the different sources of uncertainty have been identified, this work uses the following notations:

Notation in this work

- $\mathbf{e}$: Quantities to estimate (might also be referred to as epistemic)
- $\mathbf{a}$: Quantities that do not need to (or cannot) be estimated (might also be referred to as aleatory)
- Model outputs $\mathbf{y}$
- Measurements $\mathbf{y}^{exp}$

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$$\mathbf{e_{est}} = \underset{\mathbf{e}}{\mathbf{argmax}}\ \mathbf{f_{Y(E,A)|E=e}(y^{exp})} \tag{1}$$

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Note

The output is deterministic.

The Bayes formula gives:

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$$\mathbf{f}_{\mathbf{E}|\mathbf{Y}(\mathbf{E},\mathbf{A})=\mathbf{y}^{\mathbf{exp}}}(\mathbf{e}) = \frac{\mathbf{f}_{\mathbf{Y}(\mathbf{E},\mathbf{A})|\mathbf{E}=\mathbf{e}}(\mathbf{y}^{\mathbf{exp}})\mathbf{f}_{\mathbf{E}}(\mathbf{e})}{\mathbf{f}_{\mathbf{Y}(\mathbf{E},\mathbf{A})}(\mathbf{y}^{\mathbf{exp}})} \tag{2}$$

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Notation

- $\mathbf{f}_{\mathbf{E}}(\mathbf{e})$ is called the *prior*. This part can be critical.
- $\mathbf{f}_{\mathbf{E}|\mathbf{Y}(\mathbf{E},\mathbf{A})=\mathbf{y}^{\mathbf{exp}}}(\mathbf{e})$ is called the *posterior*.
- $\mathbf{f}_{\mathbf{Y}(\mathbf{E},\mathbf{A})|\mathbf{E}=\mathbf{e}}(\mathbf{y}^{\mathbf{exp}})$ is the likelihood.

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- Find the maximum
- Confidence in the estimate

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Hurdles of these approaches

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Bayes

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- Sample the posterior

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Require an accurate calculation of the likelihood.

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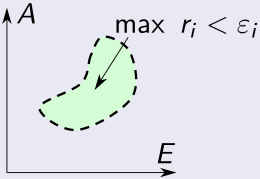
However, without assumptions

An efficient calculation of the likelihood is not straightforward.

Methodology

Fidelity Map

Big Picture



$$r_i = \left| \frac{y_i - y_i^{exp}}{y_i^{exp}} \right|$$

ε_i: Threshold

Definition

Fidelity map: part of the *space* where *all the discrepancies* r_i are *below* a reasonable threshold ε_i.

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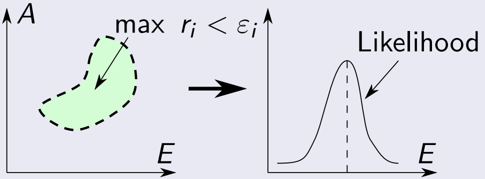

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Main Idea

Use the fidelity map in order to compute the likelihood. This definition of the fidelity map will capture the correlations (3).

Methodology

The likelihood as a probability

It can be shown that the likelihood can be expressed as:

$$\mathbf{f}_{Y(E,\mathbf{A})|E=e}(y^{exp}) = \lim_{\varepsilon \to 0} \frac{\mathbf{P}_{E=e}[y^{exp} - \varepsilon \leq Y(e,\mathbf{A}) \leq y^{exp} + \varepsilon]}{2\varepsilon}$$

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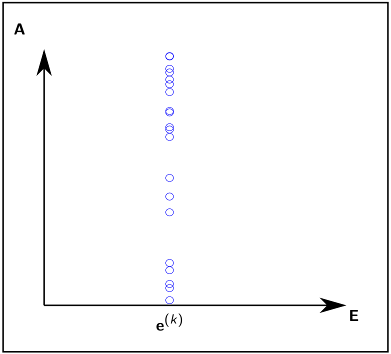

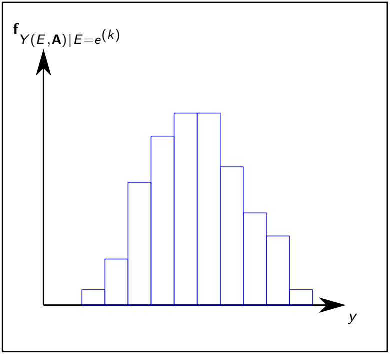

For $e^{(k)}$, propagate the uncertainties through y, according to the distribution of $\mathbf{A}$.

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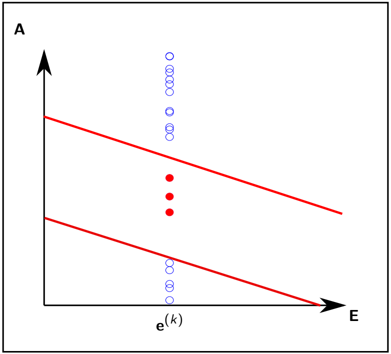

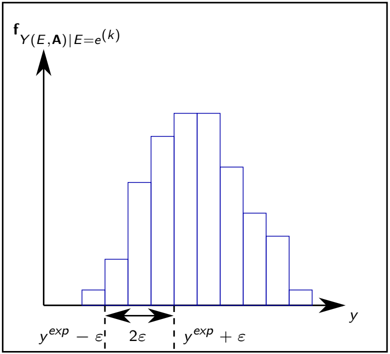

Compute the fraction of points falling into the range of 2ε.

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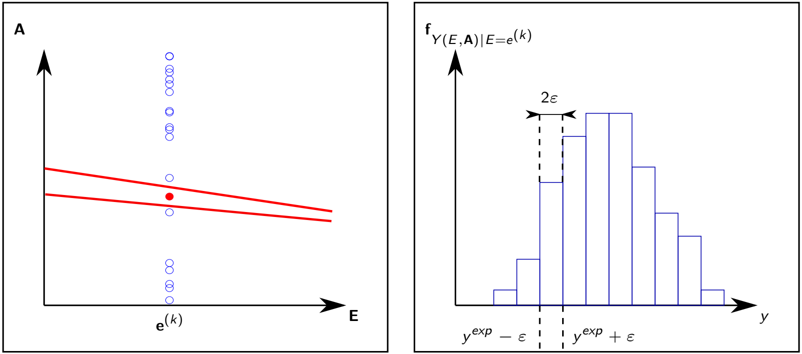

As ε tends to zero, this fraction actually becomes the likelihood value, up to a constant of 2ε.

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- Support Vector Machine can define that boundary explicitly

Methodology

Support Vector Machine (SVM) for fidelity map

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Formulation

$$s(\mathbf{x}) = b + \sum_{k=1}^{N} \lambda^{(k)} l^{(k)} K(\mathbf{x}^{(k)}, \mathbf{x})$$

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SVM properties

Only one SVM, irrespective to the number of responses ([2]). In addition, could take into account discontinuous responses.

Methodology
Adaptive Sampling

An adaptive sampling scheme[1] allows one to refine an SVM.

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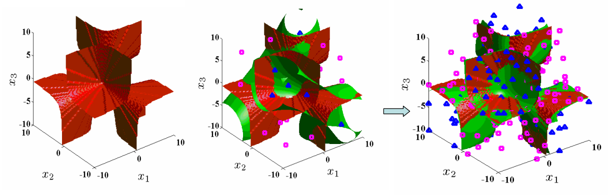

Target failure surface — Initial approximation — After adaptive sampling

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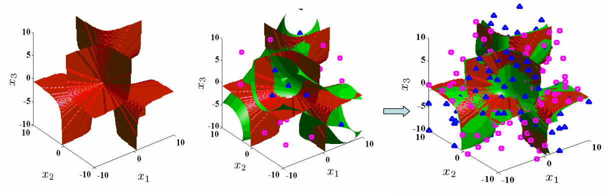

Target failure surface — Initial approximation — After adaptive sampling

Advantage

The use of the proposed adaptive sampling scheme allows to handle expensive models ([1]).
The complete computational budget is spent to build and refine the SVM.

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In summary, the presented algorithm to construct the likelihood consists of the following steps:

- Define the fidelity map using only one SVM (solve 2)
- Refine the SVM using adaptive sampling (solve 1)
- Build the likelihood using Monte Carlo sampling (solve 3)

Methodology

Likelihood estimation using SVM-based fidelity maps

In summary, the presented algorithm to construct the likelihood consists of the following steps:

- Define the fidelity map using only one SVM (solve 2)
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Monte Carlo sampling and correlations

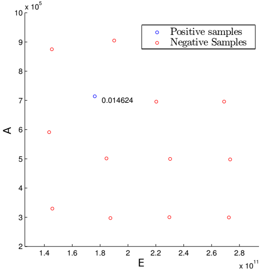
The use of Monte Carlo sampling and the definition of the fidelity map implicitly take into account any stochastic information, including correlations (3).

Methodology

Details on the process to construct the likelihood

Process

- Define a DOE
- Compute the discrepancies of these points with respect to the measurements
- Look for one point that satisfies the desired tolerances



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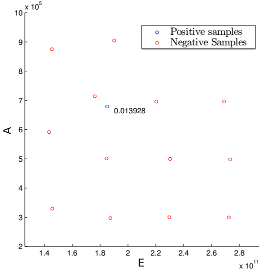

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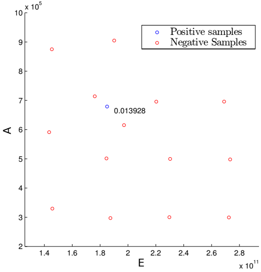

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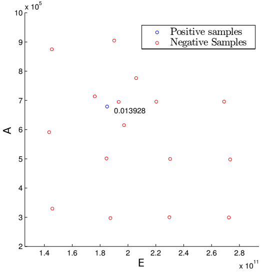

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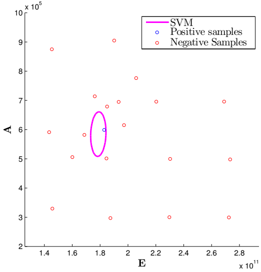
- Look for one point that satisfies the desired tolerances
- Found it



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- Refine the SVM using EDSD adaptive sampling scheme
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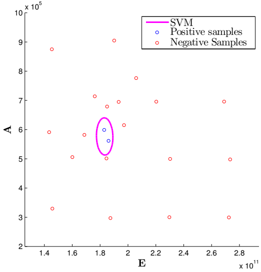

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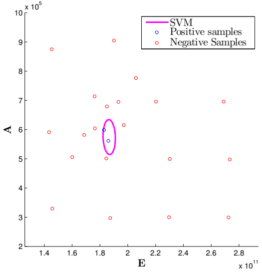

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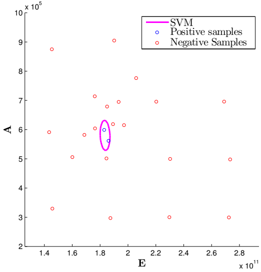

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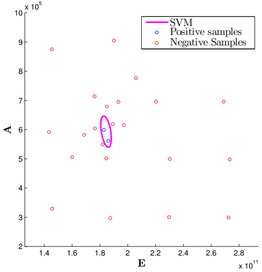

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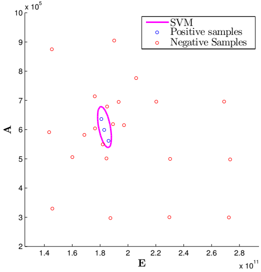

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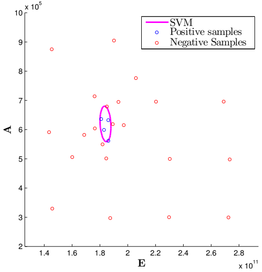

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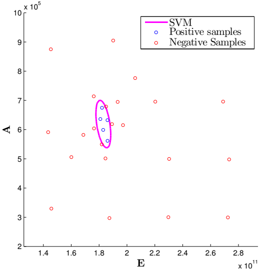

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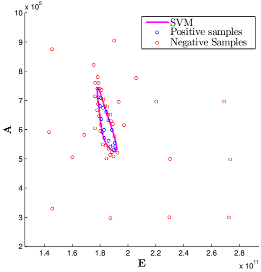
Final SVM

Begin Monte Carlo Sampling



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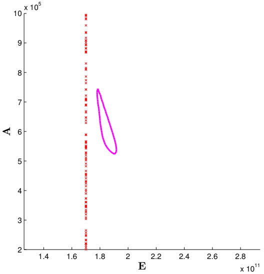
- For a given $\mathbf{e}^{(k)}$, sample the space according to the distributions of $\mathbf{A}$



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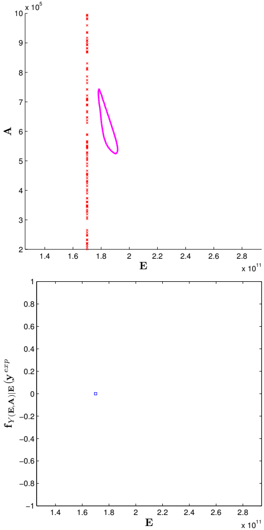

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Identification using Fidelity Maps

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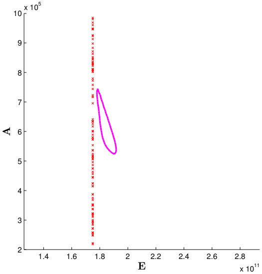

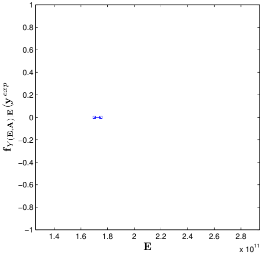

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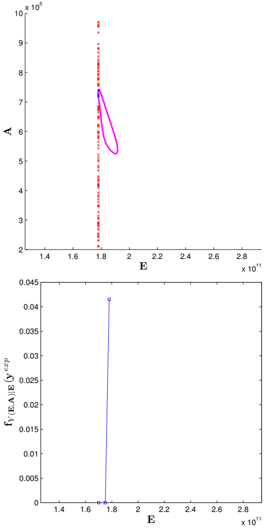

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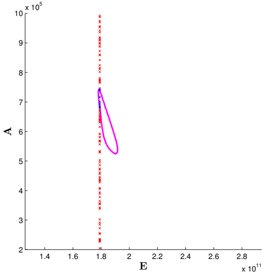

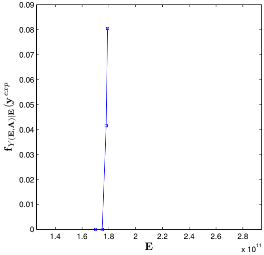

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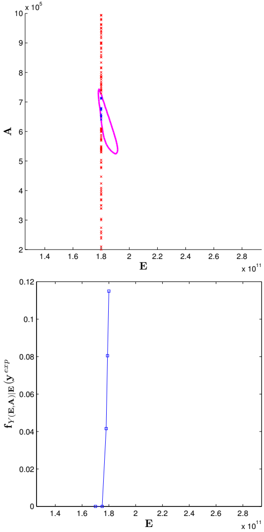

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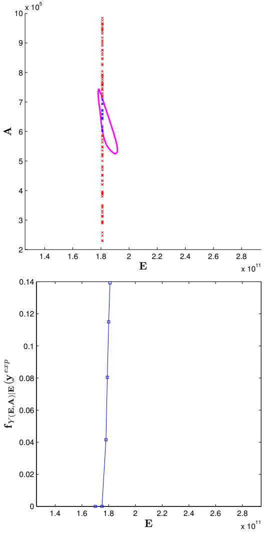

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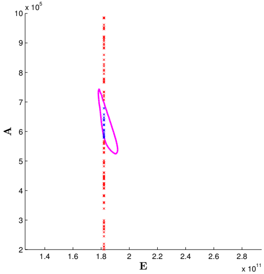

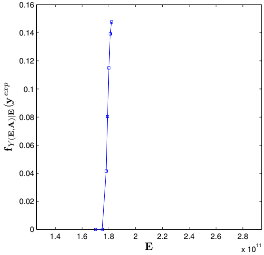

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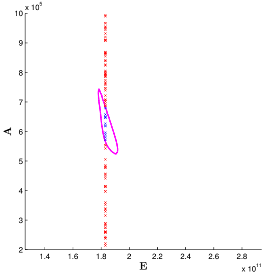

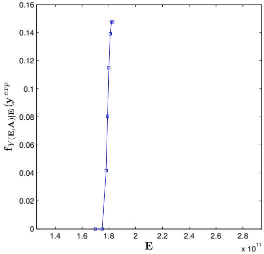

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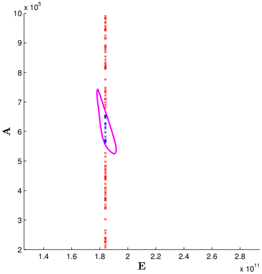

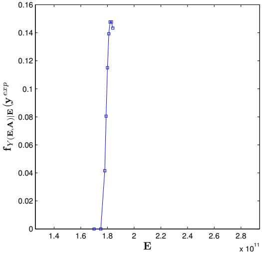

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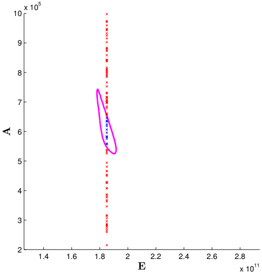

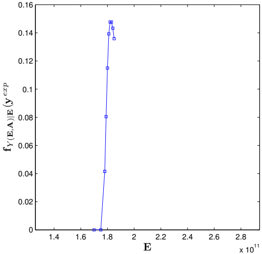

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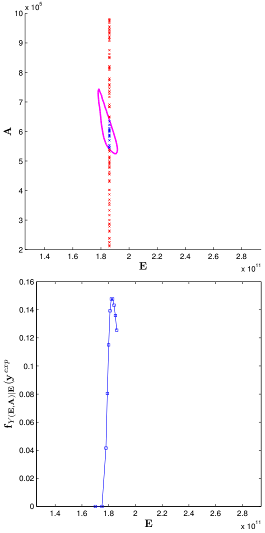

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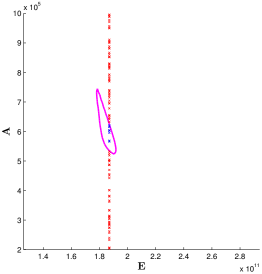

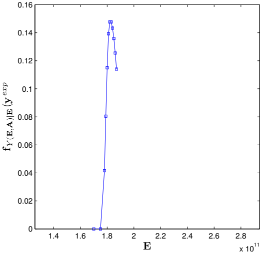

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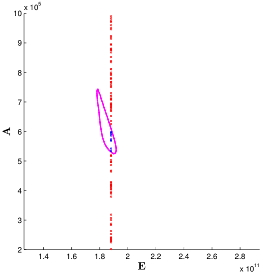

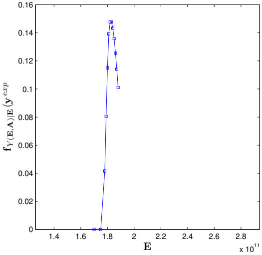

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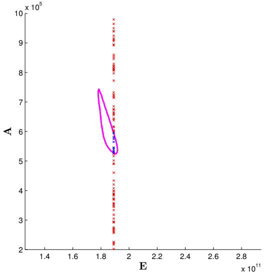

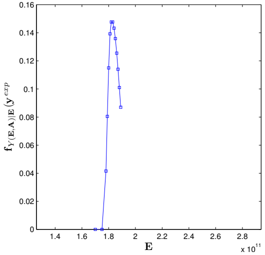

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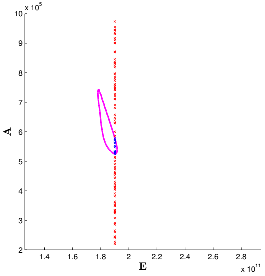

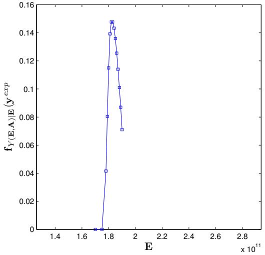

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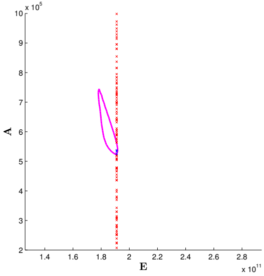

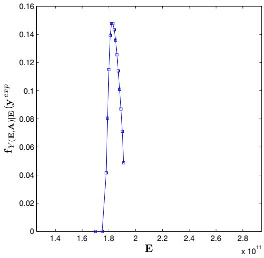

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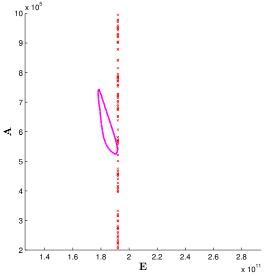

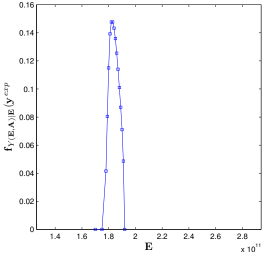

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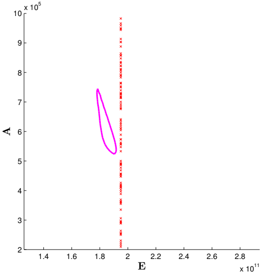

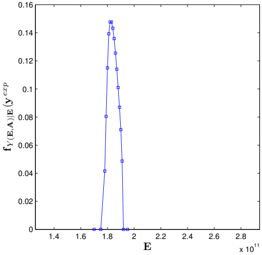

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- The MLE can now be extracted...
  ...Or the posterior can be built.

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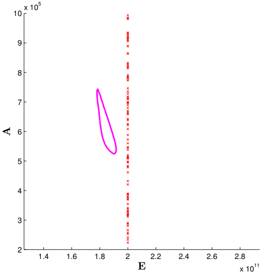

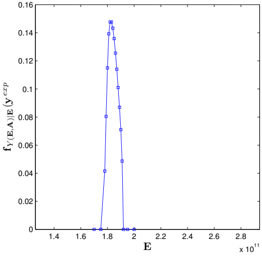

Methodology

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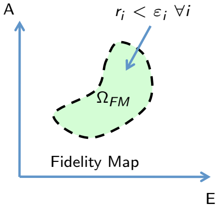

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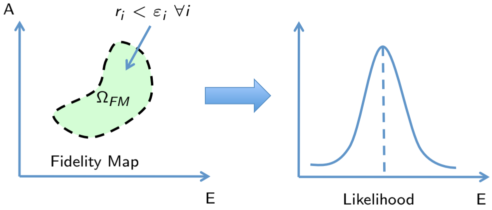

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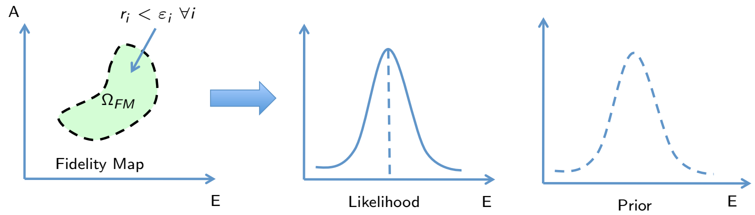

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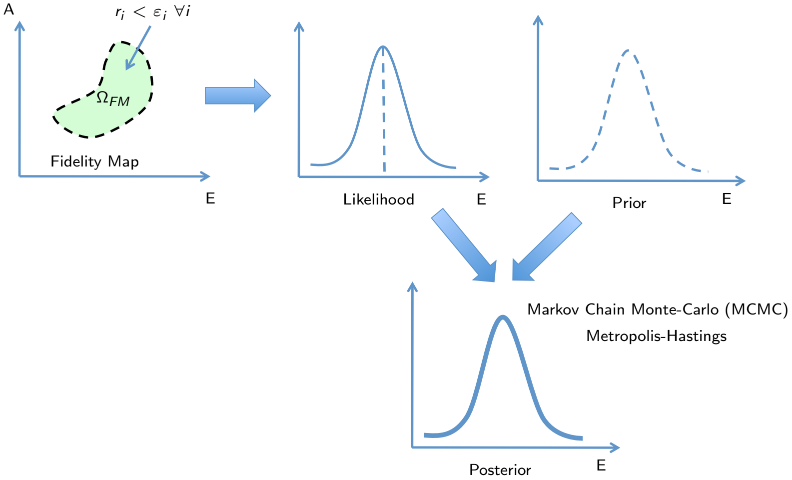

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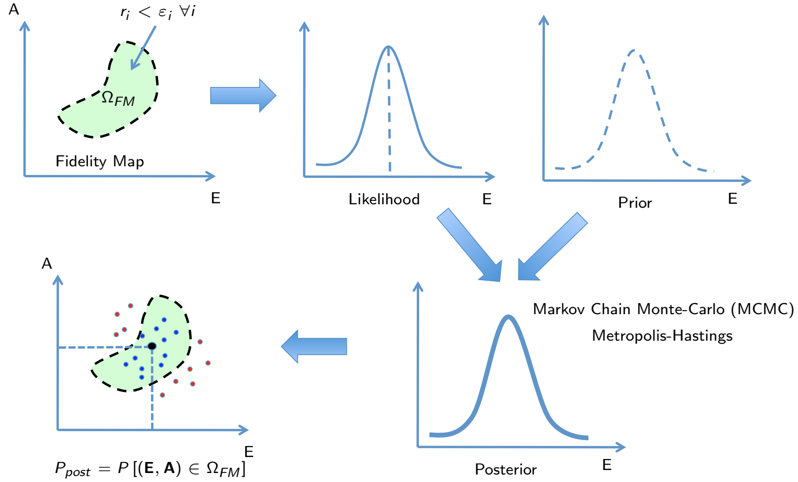

Results
Modal quantities

The methodology is now applied to two examples:

- Finite Element Model of a plate
- Finite Element Model of a piano soundboard

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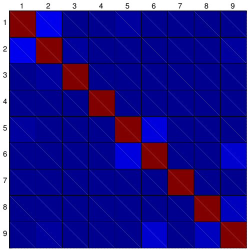
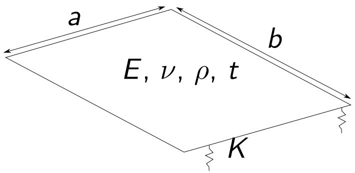
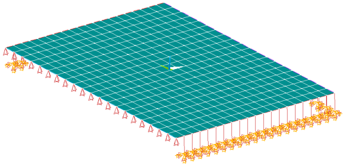


$$MAC_{ij} = \frac{(\Phi_i^{*T} A \Phi_{exp,j})^2}{(\Phi_i^{*T} A \Phi_i)(\Phi_{exp,j}^{*T} A \Phi_{exp,j})}$$

For n modes, $\dfrac{n^2+n}{2}$ terms

Results
Plate Finite Element Model





Parameters used for the plate example

Deterministic (S.I. Units)					To identify	Aleatory
a	b	ν	ρ	t	E (Pa)	K $(N.m^{-1})$
1	1.5	0.33	7800	0.01	N/A	$U(2 \times 10^5, 10^6)$

Measurements

The four first modes must be matched:

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ε

$$\varepsilon_i = 1\% \quad \forall i$$

Comparison

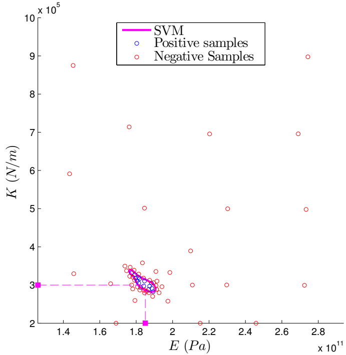
For the sake of completeness, calculate:

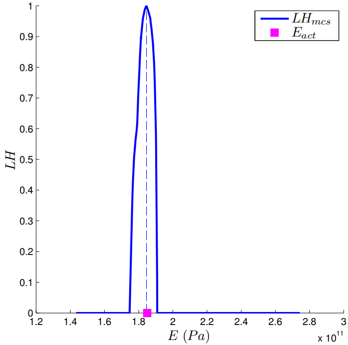
- The product of likelihoods (independence hypothesis)
- A residual-based approach:

$$R = \sum_{i=1}^{N} \left[\frac{(\lambda_i - \lambda_i^{exp})^2}{\lambda_i^{exp}} + (MAC_{ii} - 1)^2 + \sum_{j=1, j \neq i}^{N} MAC_{ij} \right]$$

Actual values

$E_{act} = 185 \times 10^9 \; Pa$ $\qquad$ $K_{act} = 3 \times 10^5 \; N.m^{-1}$





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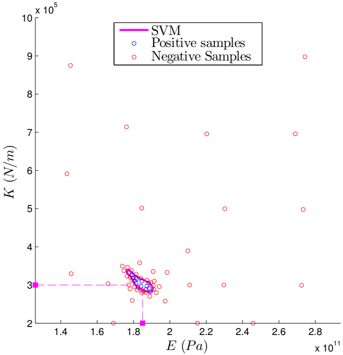
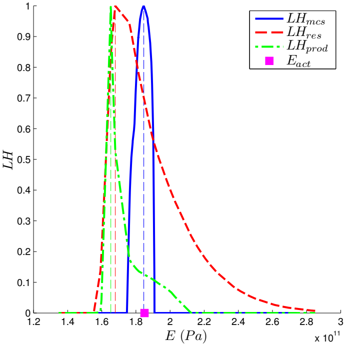
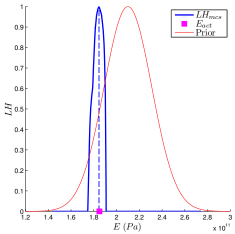




Results

Plate Finite Element Model. Posterior



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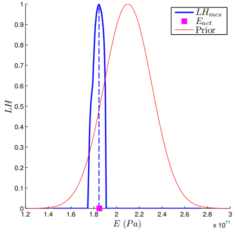

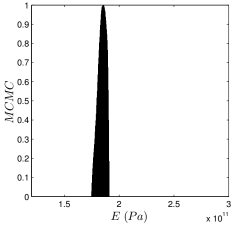

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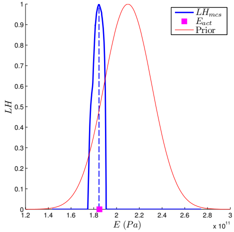

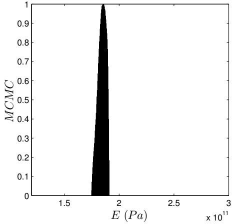

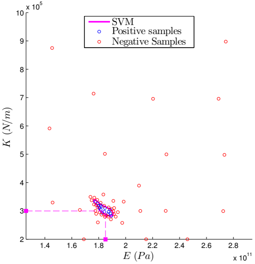

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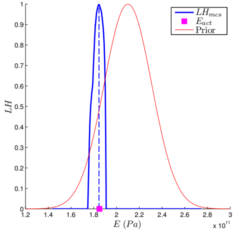

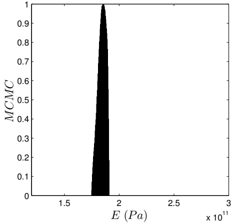

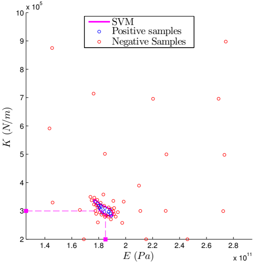

Belonging to the FM

$$P_{prior} = 0.0043$$

$$P_{post} = 0.0355$$

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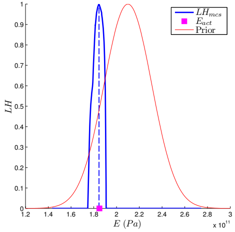

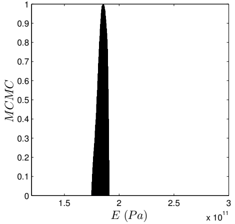

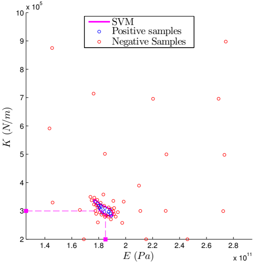

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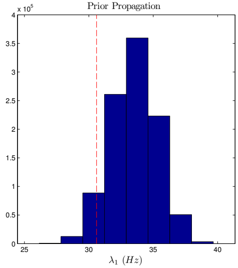
Fidelity Index

$$F_{index} = 8.2341$$

Results

Plate Finite Element Model. Propagation to response (λ_1 only)



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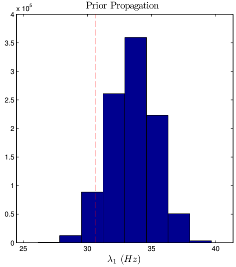

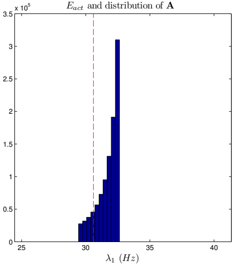

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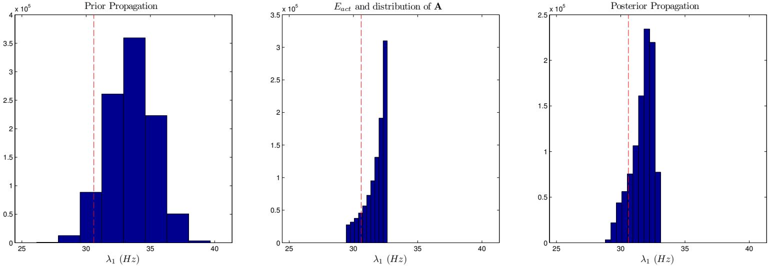

Identification using Fidelity Maps

Introduction
Motivation
Challenges
Nomenclature

Background
MLE
Bayes
Hurdles

Methodology
Fidelity Map
SVM and AS
Algo likelihood
Posterior

Results
Plate FEM
Piano FEM

Conclusion
Summary
Future work
Acknowledgement

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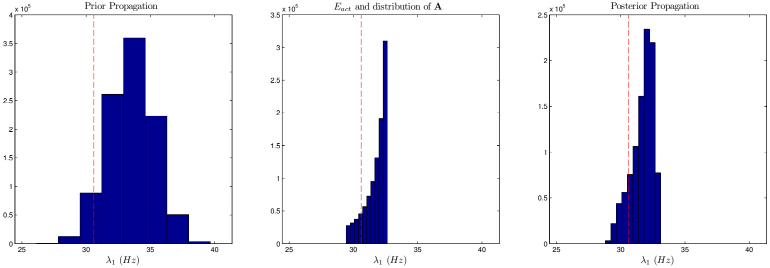
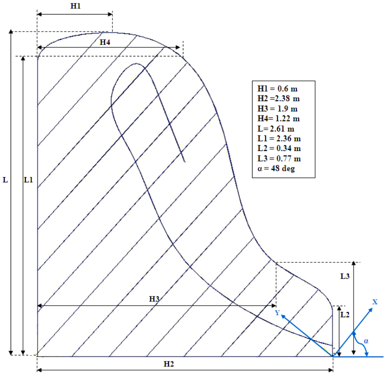
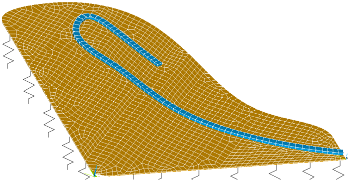


Benefits

Spread of the response has been reduced around the measurement.
Same behavior for the different responses.

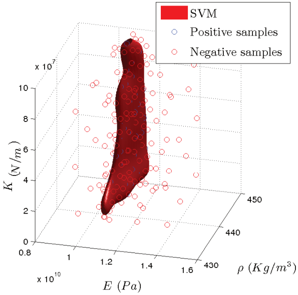
A fully parameterized model of a grand piano soundboard is used for this example. Some of the parameters are shown, along with the dimensions used for this example:

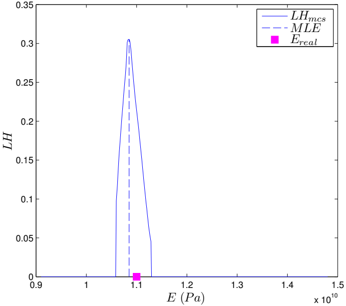




Parameters used for the piano example

To identify	Aleatory	
E_y (Pa)	ρ $(Kg.m^{-3})$	K $(N.m^{-1})$
N/A	$U(430, 450)$	$U(5 \times 10^6, 10^8)$



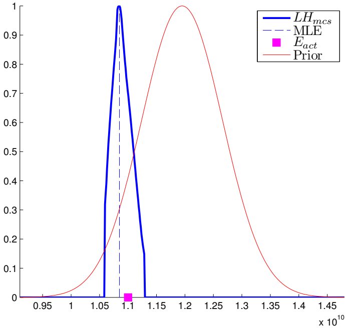


Estimated value and relative error

	E_y (Pa)	ρ $(Kg.m^{-3})$	K $(N.m^{-1})$
Actual	11×10^9	445	10^7
Estimated	10.85×10^9	N/A	N/A
Error (%)	1.36	N/A	N/A

Results

Piano Soundboard Finite Element Model. Posterior



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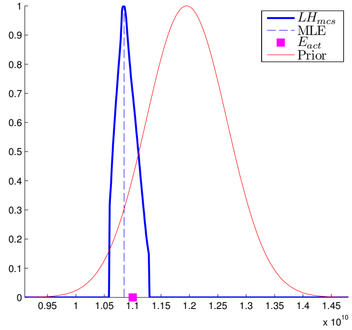

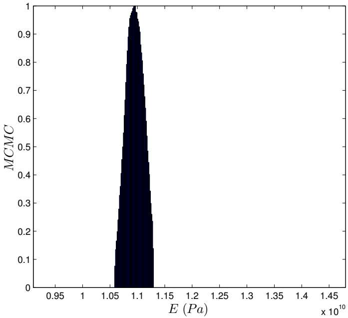

Results

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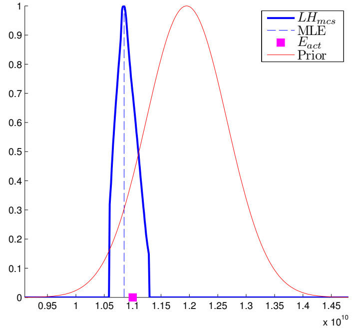
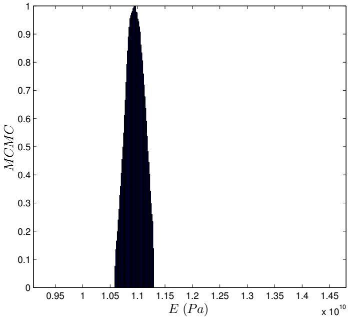




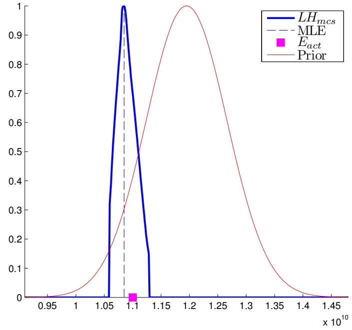
Belonging to the FM

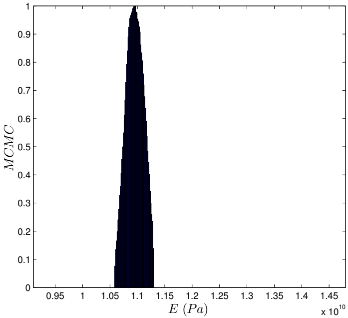
$$P_{prior} = 0.0272$$

$$P_{post} = 0.2107$$

Results

Piano Soundboard Finite Element Model. Posterior





Belonging to the FM

$$P_{prior} = 0.0272$$

$$P_{post} = 0.2107$$

Fidelity Index

$$F_{index} = 7.7388$$

Robustness

The proposed approach enables a robust estimation of the MLE.

Efficiency

- The posterior distribution can be efficiently obtained.
- Can handle a large number of responses and implicitly account for their correlations.

Conclusion

Future work

ε

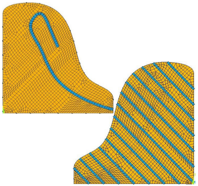
Look into ways to automatically define ε.
Speed up the process of finding an initial point satisfying ε.

Scalability

- Sensitivity analysis to reduce the dimension of the space where the SVM must be built.
- Make the approach scalable.

Acknowledgement

The authors would like to thank Ms. Fatma Mokdad for her help in the construction of the Finite Element Model of the piano soundboard.



Questions?

If you have any questions, I will be happy to answer them.

Test of the methodology

Measurements are obtained through a run of the model.

Results

Plate Finite Element Model

Test of the methodology

Measurements are obtained through a run of the model.

Robustness

The results are presented for 6 different measurements, in order to check the robustness of the proposed approach.

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Measurements are obtained through a run of the model.

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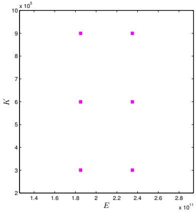
The results are presented for 6 different measurements, in order to check the robustness of the proposed approach.

E_{act} (Pa)	185×10^9		
K_{act} $(N.m^{-1})$	3×10^5	6×10^5	9×10^5
E_{act} (Pa)	235×10^9		
K_{act} $(N.m^{-1})$	3×10^5	6×10^5	9×10^5



Results

Plate Finite Element Model

Summary, $E_{act} = 185 \times 10^9 \ Pa$

Estimated values and relative errors

$K_{act} \ (N.m^{-1})$	3×10^5	6×10^5	9×10^5
$E_{est} \ (Pa)$	184.6×10^9	182.3×10^9	185.8×10^9
F_{index}	8.2341	8.8089	8.7073

Results

Plate Finite Element Model

Summary, $E_{act} = 185 \times 10^9\ Pa$

Estimated values and relative errors

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F_{index}	8.2341	8.8089	8.7073

Summary, $E_{act} = 235 \times 10^9\ Pa$

Estimated values and relative errors

$K_{act}\ (N.m^{-1})$	3×10^5	6×10^5	9×10^5
$E_{est}\ (Pa)$	232.4×10^9	237.5×10^9	235.4×10^9
F_{index}	5.5408	5.2401	7.1412

Bayesian Update
Derivation

Start with the Bayes formula:

$$\mathbf{f}_{A|B}\mathbf{f}_{B} = \mathbf{f}_{B|A}\mathbf{f}_{A} \tag{3}$$

Specializing the Bayes formula to our problem, we have:

$$\mathbf{f}_{\mathbf{E}|\mathbf{Y}(\mathbf{E},\mathbf{A})}\mathbf{f}_{\mathbf{Y}(\mathbf{E},\mathbf{A})} = \mathbf{f}_{\mathbf{Y}(\mathbf{E},\mathbf{A})|\mathbf{E}}\mathbf{f}_{\mathbf{E}} \tag{4}$$

Which can be re-written as:

$$\mathbf{f}_{\mathbf{E}|\mathbf{Y}(\mathbf{E},\mathbf{A})} = \frac{\mathbf{f}_{\mathbf{Y}(\mathbf{E},\mathbf{A})|\mathbf{E}}\mathbf{f}_{\mathbf{E}}}{\mathbf{f}_{\mathbf{Y}(\mathbf{E},\mathbf{A})}} \tag{5}$$

This is the functional expression, therefore, for a given set of parameters $\mathbf{e}$ and a given set of measurements $\mathbf{y}^{\mathbf{exp}}$:

$$\mathbf{f}_{\mathbf{E}|\mathbf{Y}(\mathbf{E},\mathbf{A})=\mathbf{y}^{\mathbf{exp}}}(\mathbf{e}) = \frac{\mathbf{f}_{\mathbf{Y}(\mathbf{E},\mathbf{A})|\mathbf{E}=\mathbf{e}}(\mathbf{y}^{\mathbf{exp}})\mathbf{f}_{\mathbf{E}}(\mathbf{e})}{\mathbf{f}_{\mathbf{Y}(\mathbf{E},\mathbf{A})}(\mathbf{y}^{\mathbf{exp}})} \tag{6}$$

Likelihood as a probability
Derivation

The relation between a PDF and a CDF is given by:

$$\mathbf{f}_{Y(E,\mathbf{A})|E=e}(y^{exp}) = \frac{\mathrm{d}\mathbf{F}_{Y(E,\mathbf{A})|E=e}}{\mathrm{d}y}(y^{exp})$$

Using Central Finite Differences, we have:

$$\lim_{\varepsilon\to 0} \frac{\mathbf{F}_{Y(E,\mathbf{A})|E=e}(y^{exp}+\varepsilon) - \mathbf{F}_{Y(E,\mathbf{A})|E=e}(y^{exp}-\varepsilon)}{2\varepsilon}$$

Wich can be re-writen as:

$$\lim_{\varepsilon\to 0} \frac{\mathbf{P}_{E=e}[Y(E,\mathbf{A}) \leq y^{exp}+\varepsilon] - \mathbf{P}_{E=e}[Y(E,\mathbf{A}) \leq y^{exp}-\varepsilon]}{2\varepsilon}$$

And finally:

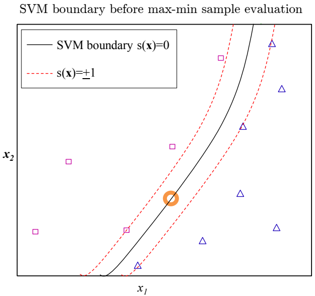
$$\mathbf{f}_{Y(E,\mathbf{A})|E=e}(y^{exp}) = \lim_{\varepsilon\to 0} \frac{\mathbf{P}_{E=e}[y^{exp}-\varepsilon \leq Y(E,\mathbf{A}) \leq y^{exp}+\varepsilon]}{2\varepsilon}$$

Adaptive sample

Two types of sample

Primary sample

$$\mathbf{x}_{mm} = \arg\max_{\mathbf{x}} \left(\min_i \|\mathbf{x} - \mathbf{x}_i\| \right), \quad \text{subject to:} \quad s(\mathbf{x}) = 0$$



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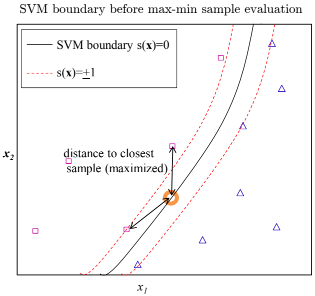

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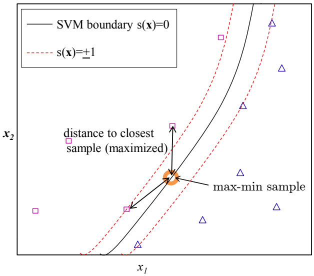
SVM boundary before max-min sample evaluation



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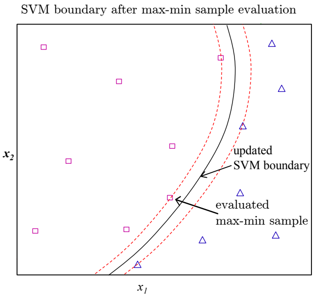


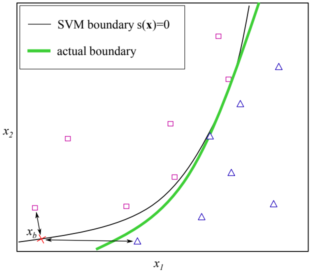
Adaptive sample

Two types of sample

Secondary sample

$$\mathbf{x}_{al} = \arg\min_{\mathbf{x}} \; s(\mathbf{x})s(\mathbf{x}_c), \quad \text{subject to:} \quad \|\mathbf{x} - \mathbf{x}_b\| \leq R$$

SVM boundary before anti-locking sample evaluation



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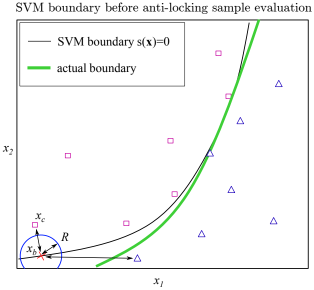

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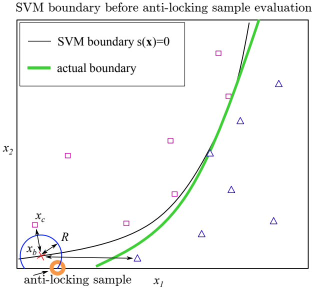

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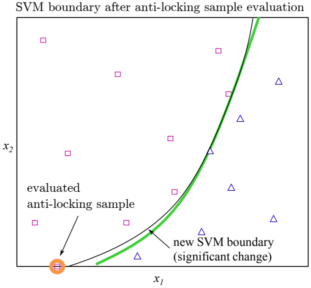

Product and Residual

Derivation

Assuming Independence

$$\mathbf{e}_{\mathbf{est}} = \underset{\mathbf{e}}{\mathbf{argmax}} \; \prod_{i=1}^{n_m} \mathbf{f}_{Y_i(\mathbf{E},\mathbf{A})|\mathbf{E}=\mathbf{e}}(y_i^{exp})$$

Reducing all outputs in one residual

A residual can be defined as:

$$R = \sum_{i=1}^{N} \left[\frac{(\lambda_i - \lambda_i^{exp})^2}{\lambda_i^{exp}} + (MAC_{ii} - 1)^2 + \sum_{j=1, j \neq i}^{N} MAC_{ij} \right]$$

Then an MLE can be found using:

$$\mathbf{e}_{\mathbf{est}} = \underset{\mathbf{e}}{\mathbf{argmax}} \; \mathbf{f}_{R(\mathbf{E},\mathbf{A})|\mathbf{E}=\mathbf{e}}(0)$$

Kernel Smoothing
Basic notion

For a given kernel K symmetric around x_c and satisfying $\int K(x_c, x)dx = 1$, it is possible to approximate the PDF used to shoot a sample of n points $\mathbf{x}$ as:

$$PDF_X(x) = \frac{1}{n}\sum_{i=1}^{n} K(x_i, x)$$

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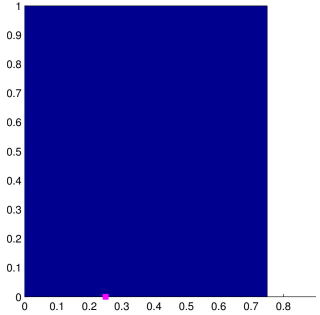

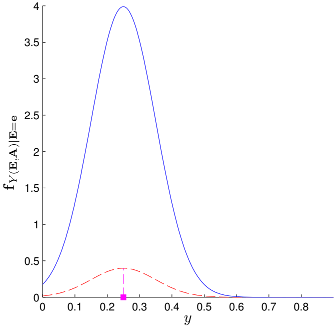

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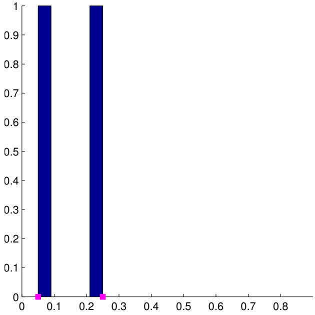

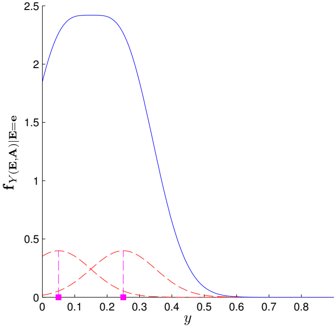

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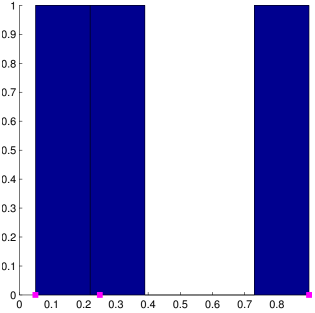

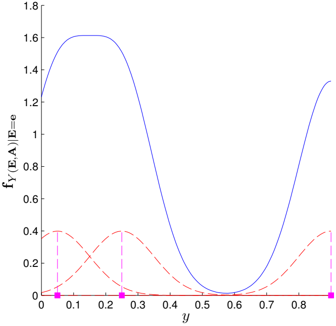

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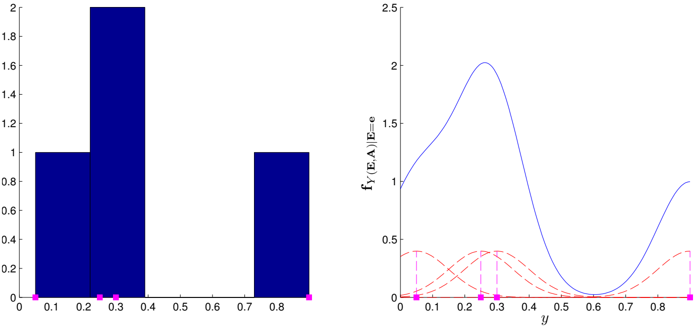

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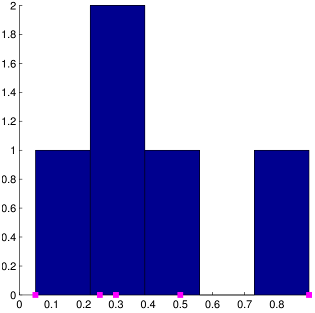

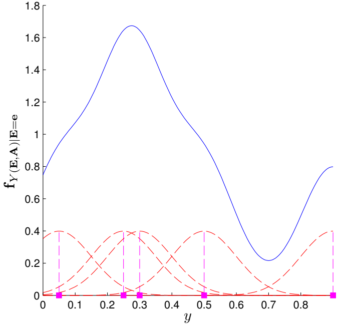

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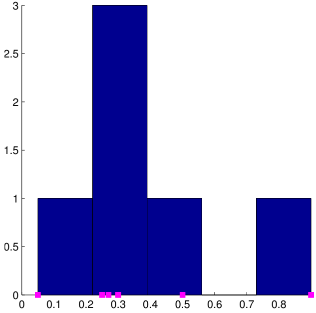

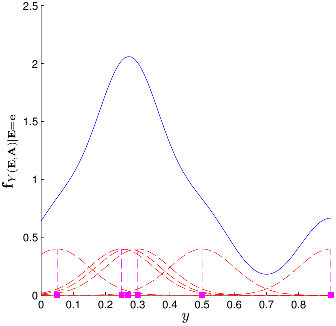

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